MATH 665 PROBLEM SET 3

FALL 2024

Due Thursday, November 14. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let G be a connected smooth reductive algebraic group over $\bar{\mathbf{F}}_q$. Fix a Frobenius map F corresponding to an $\mathbf{F}_q$-form and an F-stable Borel pair (B, T). Assume that F acts trivially on the Weyl group W. Recall that

$$\begin{aligned} H_{T^F}^{G^F}(1) &:= \mathrm{End}_{AG^F}(\mathrm{Ind}_{AB^F}^{AG^F}(1)) \qquad \text{where } A = \mathbf{Z}[q^{\pm 1/2}] \\ &\simeq H_W(\mathsf{x})|_{\mathsf{x}\to q^{1/2}}. \end{aligned}$$

Under the isomorphism, $h_w \in H_{T^F}^{G^F}(1)$ corresponds to $q^{\ell(w)/2}\sigma_w$.

(1) For $w, w', w'' \in W$ and $x, y \in G^F$ such that $yB \xrightarrow{w} xB$, express

$$|\{zB^F \in G^F/B^F \mid yB \xrightarrow{w'} zB \xrightarrow{w''} xB\}|$$

in terms of $h_w, h_{w'}, h_{w''}$.

(2) Take $G = \mathrm{PGL}_2$, so that $W = \{e, s\}$. Use (1) to express $P_m(q)$ in terms of h_s^m and G^F, where the P_m are the polynomials from Problem Set 2.

For Problems 2–3, we need the diagrammatic presentation of the Temperley–Lieb algebra TL_n. The figure to the right depicts the standard $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-linear basis of TL_3 in the order $1, e_1, e_2, e_1e_2, e_2e_1$ from top to bottom.

Problem 2. For any $w \in S_n$, written as a word $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of minimal length, let

$$e_w = e_{i_1}e_{i_2}\cdots e_{i_\ell} \in TL_n.$$

By Fan–Green, $w \mapsto e_w$ is a bijection from the set of 321-avoiding permutations in S_n onto the standard basis of TL_n. Using this result:

(1) List the standard basis elements for TL_4.

(2) Using the diagrammatics, show that the rank of TL_n is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Alternatively, show that this number counts the 321-avoiding permutations of n objects.

Problem 3. From the diagram for a basis element e, we can form the *annular closure* $\hat{e}$, analogous to the link closure of a braid. Let $\nu_n : TL_n \to \mathbf{Z}[\mathsf{x}^{\pm 1}]$ be the $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-linear trace defined on the standard basis by

$$\nu_n(e) = (\mathsf{x} + \mathsf{x}^{-1})^{\bigcirc(e)},$$

where $\bigcirc(e)$ is the number of connected components of $\hat{e}$.

(1) For $f \in TL_{n-1}$, how are $\nu_n(f)$ and $\nu_n(fe_n)$ related to $\nu_{n-1}(f)$?

(2) Let $j_n : H_{S_n} \to TL_n$ be the $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-algebra homomorphism that sends $c_w \mapsto e_w$. Show that the composition

$$\tilde{\nu}_n : H_{S_n} \xrightarrow{j_n} TL_n \xrightarrow{\nu_n} \mathbf{Z}[\mathsf{x}^{\pm 1}]$$

is a specialization of $(\mathsf{x} + \mathsf{x}^{-1})\mu_n$, where $\mu_n : H_{S_n} \to \mathbf{Z}[\mathsf{x}^{\pm 1}, \frac{1}{\mathsf{x}-\mathsf{x}^{-1}}][\mathsf{a}^{\pm 1}]$ is the HOMFLYPT Markov trace.

(3) Repeat (2) with the homomorphism that sends $\kappa(c_w) \mapsto -e_w$ in place of j_n. How does the specialization change?

Extra: For 321-avoiding $w \in S_n$, how is $\bigcirc(e_w)$ related to w?

Problem 4. For $n = 2, 3$, the character tables below list values of the characters $\chi_{\mathsf{x}} : H_{S_n} \to \mathbf{Z}[\mathsf{x}^{\pm 1}]$ that correspond to the irreducible characters $\chi : S_n \to \mathbf{Z}$ under Tits's deformation theorem. Recall that we write $\sigma_i = \sigma_{s_i}$.

$n = 2:$

	1	σ_1
1	1	x
ε	1	$-\mathsf{x}^{-1}$

$n = 3:$

	1	σ_1	$\sigma_{s_1 s_2} = \sigma_1\sigma_2$
1	1	x	x^2
φ	2	$\mathsf{x} - \mathsf{x}^{-1}$	-1
ε	1	$-\mathsf{x}^{-1}$	x^{-2}

(1) Why don't we need a column for $\sigma_{s_1 s_2 s_1} = \sigma_1\sigma_2\sigma_1$ in the $n = 3$ table?

(2) Decompose the Markov traces μ_2 and μ_3 into $\mathbf{Q}(\mathsf{a}, \mathsf{x})$-linear combinations of irreducible Hecke characters.

(3) Repeat the $n = 3$ case of (2) with $\tilde{\nu}_3$ in place of μ_3. Some weight in the linear combination will now vanish. Which one?

Extra: Look up the reduced Burau representation of Br_3. Using the $n = 3$ table, show that it factors through $H_{S_3}^{\mathsf{x}}$.

In Problem 5, we use the notation and conventions for Soergel bimodules from the "active learning" notes for 10/22. Specifically, we take $W = S_2 = \{e, s\}$.

Problem 5. We will discover how to simplify iterated convolutions of $\mathcal{R}_s^+$ up to homotopy of complexes of graded R-bimodules.

(1) Let $\Phi : \mathbf{B}_s\langle 1\rangle \oplus \mathbf{B}_s\langle -1\rangle \xrightarrow{\sim} \mathbf{B}_s \otimes \mathbf{B}_s$ be the explicit isomorphism from the notes. Check that the composition

$$\mathbf{B}_s\langle 1\rangle \oplus \mathbf{B}_s\langle -1\rangle \xrightarrow{\Phi} \mathbf{B}_s \otimes \mathbf{B}_s \xrightarrow{(\epsilon\otimes\mathrm{id}, \mathrm{id}\otimes\epsilon)} \mathbf{B}_s\langle 1\rangle \oplus \mathbf{B}_s\langle 1\rangle$$

sends $(1 \otimes 1, 0) \mapsto (1 \otimes 1, 1 \otimes 1)$ and $(0, 1 \otimes 1) \mapsto \frac{1}{2}(\alpha \otimes 1, 1 \otimes \alpha)$.

(2) Use (1) to show that $\mathcal{R}_s^+ * \mathcal{R}_s^+$ is homotopic to

$$\underline{\mathbf{B}_s\langle -1\rangle} \xrightarrow{\frac{1}{2}(\mathrm{id}\otimes\alpha - \alpha\otimes\mathrm{id})} \mathbf{B}_s\langle 1\rangle \xrightarrow{\epsilon} \mathbf{B}_e\langle 2\rangle.$$

Similarly, show that $\mathcal{R}_s^+ * \mathcal{R}_s^+ * \mathcal{R}_s^+$ is homotopic to

$$\underline{\mathbf{B}_s\langle -2\rangle} \xrightarrow{\frac{1}{2}(\mathrm{id}\otimes\alpha + \alpha\otimes\mathrm{id})} \mathbf{B}_s \xrightarrow{\frac{1}{2}(\mathrm{id}\otimes\alpha - \alpha\otimes\mathrm{id})} \mathbf{B}_s\langle 2\rangle \xrightarrow{\epsilon} \mathbf{B}_e\langle 3\rangle.$$