MATH 665 PROBLEM SET 2

FALL 2024

Due Thursday, October 17. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let G be a (connected, smooth) reductive algebraic group over k with Frobenius F, compatible with the group law. Let (B, T) be an F-stable Borel pair. For $w \in N_G(T)/T$ lifting to $\dot{w} \in N_G(T)$, recall the variety

$$\tilde{X}_{\dot{w}} = \{g \in G \mid g^{-1}F(g) \in U\dot{w}U\}/U, \qquad \text{where } U = [B, B].$$

For any F-stable maximal torus $T' \subseteq G$ and Borel $B' \subseteq G$ containing T', let

$$\tilde{X}_{U'} = \{h \in G \mid h^{-1}F(h) \in F(U')\}/(U' \cap F(U')), \quad \text{where } U' = [B', B'].$$

Show that for fixed T', we can exhibit $B', w, \dot{w}$, and

$$f : \tilde{X}_{U'} \to \tilde{X}_{\dot{w}}$$

such that:

(1) $(T')^F \simeq T^{wF}$.
(2) f is bijective on k-points.
(3) f is G^F-equivariant, and transports the right $(T')^F$-action on $\tilde{X}_{U'}$ to the right T^{wF}-action on $\tilde{X}_{\dot{w}}$.

Hint: Consider the intermediate variety $\{h \in G \mid h^{-1}F(h) \in F(U')\}/U'$.

Extra: If you are bored, show that f is in fact an isomorphism of sheaves in the smooth topology, hence of varieties.

This problem suggests how, given an F-stable maximal torus $T' \subseteq G$ of type w and character $\theta' : (T')^F \to \bar{\mathbf{Q}}_\ell^\times$, we can construct a class function $R_{T',\theta'}$ directly from T' and θ' such that $R_{T',\theta'} = R_{T,\theta}$ for some $\theta : T^{wF} \to \bar{\mathbf{Q}}_\ell^\times$.

Problem 2. Fix an integer $n > 0$, invertible in $k = \bar{\mathbf{F}}_q$. Let $\mathcal{B}$ be the variety of Borel subgroups of GL_n, which we identify with the variety of complete flags in $\mathbf{A}^n$. Below, we work with the standard Frobenius on $\mathcal{B}$, induced by the map F on $\mathbf{A}^n$ that sends $F(x_i) = x_i^q$ on coordinates.

Recall that if a pair of flags $(\vec{V}, \vec{V}')$ corresponds to a pair of Borels in relative position $w \in S_n$, then k^n admits a basis $e_1, \ldots e_n$ such that $V_i = \langle e_1, \ldots, e_i\rangle$ and $V_i' = \langle e_{w(1)}, \ldots, e_{w(n)}\rangle$. Consider the n-cycle

$$c = (1, 2, \ldots, n) \in S_n.$$

(1) Identifying the Deligne–Lusztig variety X_c with a subvariety of $\mathcal{B}$, write the defining condition on X_c in terms of flags.

(2) Show that the map $\vec{V} \mapsto V_1 : \mathcal{B} \to \mathbf{P}^n$ restricts to a bijection

$$X_c(k) \xrightarrow{\sim} \{L \in \mathbf{P}^n(k) \mid L + F(L) + \cdots + F^{n-1}(L) = k^n\}.$$

Deduce that X_c is the nonvanishing locus of an explicit determinant.

(3) Can you guess a similar description for $\tilde{X}_{\dot{c}}$? *Hint:* Look back at #4 on Problem Set 1.

Problem 3. Let A be the localization of $\mathbf{Z}[\mathsf{x}]$ at the prime $\mathsf{x} - 1$. Let

$$H_2 = \frac{A[\sigma]}{\langle \sigma^2 - (\mathsf{x} - \mathsf{x}^{-1})\sigma - 1 \rangle},$$

viewed as the Hecke algebra of S_2 over A.

(1) Show that the ring isomorphism $H_2/(\mathsf{x} - 1) \xrightarrow{\sim} \mathbf{Z}S_2$ lifts to an A-*algebra isomorphism* $H_2 \xrightarrow{\sim} AS_2$.

(2) Bootstrap[1] (1) to show the analogous result with

$$H_3 = \frac{A[\sigma_1, \sigma_2]}{\langle \sigma_1^2 - (\mathsf{x} - \mathsf{x}^{-1})\sigma_1 - 1,\ \sigma_2^2 - (\mathsf{x} - \mathsf{x}^{-1})\sigma_2 - 1,\ \sigma_1\sigma_2\sigma - \sigma_2\sigma_1\sigma_2 \rangle}$$

and S_3 in place of H_2 and S_2.

(3) For $n = 2, 3$, determine the elements of H_n that correspond, under the A-algebra isomorphisms above, to the central idempotents $\sum_{w \in S_n} \chi(w)w$ with $\chi \in \mathrm{Irr}(S_n)$.

In the problems that follow, $\mathbf{P}$ denotes the reduced HOMFLY-PT polynomial in variables a and x.

Problem 4. Let $\beta_1 = \sigma_1^3 \in Br_2$ and $\beta_2 = (\sigma_1\sigma_2)^2 \in Br_3$.

(1) Using only pictures, show that the link closures $\hat{\beta}_1$ and $\hat{\beta}_2$ are isotopic.

(2) Using only the Jones–Ocneanu construction of $\mathbf{P}$ from Markov traces, show that $\mathbf{P}(\hat{\beta}_1) = \mathbf{P}(\hat{\beta}_2)$.

(3) Prove that $\hat{\beta}_1, \widehat{\beta_1^{-1}}$ cannot be isotopic.

Problem 5. Let σ_1 denote the positive generator of Br_2.

(1) Show that the polynomials $\mathbf{P}(\widehat{\sigma_1^n})$ satisfy a linear recurrence for $n \geq 0$.

(2) Deduce that the (positive) $(2, n)$ torus links are pairwise non-isotopic.

Problem 6. Consider the varieties

$$\mathcal{X}_m = \{\vec{L} = (L_1, \ldots, L_m) \in (\mathbf{P}^1)^m \mid L_{i-1} \neq L_i \text{ for all } i \bmod m\},$$

which can be defined over any field. Show that:

(1) For each $m \geq 2$, there is a polynomial $P_m(\mathsf{q}) \in \mathbf{Z}[\mathsf{q}]$ satisfying $P_m(q) = |\mathcal{X}_m(\mathbf{F}_q)|$ for all (odd) prime powers q.

Hint: Partition $\mathcal{X}_m$ into subvarieties $\mathcal{X}_{\vec{e}}$ indexed by binary sequences $\vec{e} \in \{0, 1\}^{m-1}$, where $\mathcal{X}_{\vec{e}}$ consists of the tuples $\vec{L}$ in which $L_i = L_m$ if and only if $e_i = 0$.

(2) The polynomials $P_m(\mathsf{q})$ satisfy a linear recurrence. Can you guess how they are related to the polynomials $\mathbf{P}(\widehat{\sigma_1^n})$ in Problem 5?

[1] **(10/7)**: "Bootstrap" is not such an accurate description—thank-you to River for catching.