MATH 665 PROBLEM SET 1

FALL 2024

Due Thursday, September 19. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 9/14, in blue.**

Problem 1.
(1) Compute the sizes of the conjugacy classes of S_4, S_5, S_6.
(2) Use (1) to show that A_4 is not simple, but A_5 and A_6 are.

Problem 2.
(1) Show that $|\mathrm{SL}_2(\mathbf{F}_7)| = 2 \cdot 3 \cdot 7 \cdot 8$.
(2) Find a reference listing the eleven conjugacy classes of $\mathrm{SL}_2(\mathbf{F}_7)$.
(3) Use (2) to compute the six conjugacy classes of

$$\mathrm{PSL}_2(\mathbf{F}_7) = \mathrm{SL}_2(\mathbf{F}_7)/\{\pm 1\}$$

and their sizes.
(4) Use (1) and (3) to show that $\mathrm{PSL}_2(\mathbf{F}_7)$ is simple.

Problem 3. Over $\bar{\mathbf{F}}_q$, for q odd, let $G = \mathrm{SL}_2$. Let $B = TU$ be its upper-triangular subgroup, where T is the diagonal torus and U the unipotent radical of B. Let $F : G \to G$ correspond to the split $\mathbf{F}_q$-form, so that B, T, U are F-stable. For any character χ of T^F, viewed as a character of B^F, let $I_\chi = \mathrm{Ind}_{B^F}^{G_F}(\chi)$.

(1) Taking $q = 3$:
   (a) Use Bruhat to find the number of double cosets of U^F in G^F.
   (b) For all χ, use Mackey to decompose I_χ into its irreducible summands as a representation of G^F. The total number of χ, times $|W^F|$, should match your answer to (a).
(2) Repeat (2), now taking $q = 5$.

Problem 4. Keep the setup of the previous problem. Recall the Deligne–Lusztig variety

$$\tilde{X}_s = \{gU \in G/U \mid g^{-1}F(g) \in U\dot{s}U\}, \quad \text{where } \dot{s} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

The G-action on $\mathbf{A}^2$ induces an isomorphism $G/U \xrightarrow{\sim} \mathbf{A}^2 \setminus \{0\}$. Show that at the level of $\bar{\mathbf{F}}_q$-points, this isomorphism identifies $\tilde{X}_s$ with the plane curve $xy^q - x^q y = 1$, where x, y are the standard coordinates on $\mathbf{A}^2$.

Problem 5. Let q be any prime power. Over $\bar{\mathbf{F}}_q$, let X be an algebraic variety with an action of a smooth algebraic group H. Suppose that there are Frobenius maps F on X and H such that $F(h \cdot x) = F(h) \cdot F(x)$. Show that:

(1) If H is *connected*, then every F-stable $H(\bar{\mathbf{F}}_q)$-orbit on $X(\bar{\mathbf{F}}_q)$ has an F-fixed point. *Hint:* Pick a point and apply Lang's theorem.

(2) In the setting of (1), suppose that the action map

$$\begin{aligned} H \times X &\to X \times X \\ (h, x) &\mapsto (h \cdot x, x) \end{aligned}$$

is smooth and its pullback along the diagonal $X \to X \times X$ has connected geometric fibers. Note that the fibers of this pullback are the stabilizers of the H-action on X. Deduce that there is a bijection $(X/H)^F \simeq X^F/H^F$.

(3) If H is not connected, then the conclusions to (1)–(2) fail, even when $X = \mathbf{A}^1 \setminus \{0\}$ and H acts freely.

Problem 6. Over an algebraically closed field k of characteristic not 2, let $Z \subseteq \mathrm{GL}_2$ be the subgroup of scalar matrices, acting on the larger group by multiplication.

(1) Compute the subring $k[\mathrm{GL}_2]^Z \subseteq k[\mathrm{GL}_2]$.

(2) Deduce that the embedding $\mathrm{SL}_2 \to \mathrm{GL}_2$ descends to an isomorphism

$$\mathrm{GL}_2 /\!\!/ Z \xrightarrow{\sim} \mathrm{SL}_2 /\!\!/ \{\pm 1\}.$$

Above, $k[X /\!\!/ H] := k[X]^H$ for any algebraic variety X over k with an action of an algebraic group H.

This problem suggests why we prefer not to define an algebraic group PSL_2 distinct from PGL_2.[1]

[1]See https://mathoverflow.net/a/16150 for further context.