13. (Active Learning)

Today is an interlude about a purely algebraic approach to categorifying the Iwahori–Hecke algebra. We will hint at how this approach is related to geometry, but defer the full details to a later lecture. The notes that follow take the form of an open-ended problem set in the ROSS/PROMYS style.

Fix a field K of characteristic zero. Given a $\mathbf{Z}$-graded K-vector space $V = \bigoplus_i V^i$, we write $V\langle n\rangle$ for the graded vector space in which

$$V\langle n\rangle^i = V^{i+n}.$$

13.1.

We start with $G = \mathrm{SL}_2$ and $W = \{e, s\}$. We will introduce a graded ring R and a W-action on R motivated by the geometry of G, though the geometry is not needed for the problems that follow.

Recall that the diagonal torus of G is a copy of GL_1, and that W acts on it according to $s \cdot z = z^{-1}$. Note that $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$. The resulting W-action on $\mathbf{C}^\times$ induces a W-action on the homotopy type of the classifying space $[pt/\mathbf{C}^\times]$. The singular cohomology of the latter with coefficients in K is

$$R := \mathrm{H}^*([pt/\mathbf{C}^\times], K) = K[\alpha], \quad \text{where } \deg \alpha = 2,$$

and the induced W-action on R is given by

$$s \cdot \alpha = -\alpha.$$

Thus $R^s = K[\alpha^2]$.

Problem 13.1. Consider the formula

$$\partial(f) = \tfrac{1}{\alpha}(f - s \cdot f).$$

(1) Show that ∂ is a well-defined operator on R such that

$$\partial^2 = 0 \quad \text{and} \quad \partial(f_1 f_2) = \partial(f_1) f_2 + (s \cdot f_1)\partial(f_2).$$

This explains the notation ∂.

(2) Use ∂ to write down an explicit isomorphism of graded R^s-bimodules

$$R \xrightarrow{\sim} R^s \oplus R^s\langle -2\rangle.$$

Hint: Interpret ∂ as a grading-preserving morphism of R^s-bimodules $\partial : R \to R^s\langle -2\rangle$. Show that it is surjective with kernel R^s, then find an explicit splitting.

Problem 13.2. Consider the graded R-bimodules

$$\mathbf{B}_e = R \quad \text{and} \quad \mathbf{B}_s = R \otimes_{R^s} R\langle 1\rangle.$$

(1) Use the previous problem to check that $\mathbf{B}_s \simeq R\langle 1\rangle \oplus R\langle -1\rangle$ as either graded *left* R-modules or graded *right* R-modules.
(2) Observe that $b_e = 1 \otimes 1$ and $b_s = \frac{1}{2}(\alpha \otimes 1 + 1 \otimes \alpha)$ form homogeneous elements of $\mathbf{B}_s$. Which degrees do they occupy? Show that

$$\begin{aligned} f b_e &= b_e(s \cdot f) + b_s \partial(f), \\ b_e f &= (s \cdot f) b_e + \partial(f) b_s, \\ f b_s &= b_s f \end{aligned}$$

for all $f \in R$. Deduce that $\mathbf{B}_s \not\simeq R\langle 1\rangle \oplus R\langle -1\rangle$ as *R-bimodules*.

Problem 13.3. Identify $\mathbf{B}_s \otimes_R \mathbf{B}_s = R \otimes_{R^s} R \otimes_{R^s} R\langle 2\rangle$.

(1) Use the previous problem to check that $\mathbf{B}_s \otimes_R \mathbf{B}_s \simeq \mathbf{B}_s\langle 1\rangle \oplus \mathbf{B}_s\langle -1\rangle$ as graded R-bimodules.
(2) Deduce that the graded additive category C_W generated by $\mathbf{B}_e$ and $\mathbf{B}_s$ under direct sums and grading shifts is closed under $\otimes_R$.
   Further deduce that the split Grothendieck group $[\mathsf{C}_W]_\oplus$, equipped with the product induced by $\otimes_R$, is isomorphic to the Hecke algebra of $W = S_2$ over $\mathbf{Z}[\mathsf{x}^{\pm 1}]$. Where do $\mathbf{B}_e$ and $\mathbf{B}_s$ go?

Extra: We can make the isomorphism in (1) explicit.

Let μ, δ be the (grading-preserving) morphisms of R-bimodules determined by the formulas below:

$$\begin{aligned} \mathbf{B}_s \otimes_R \mathbf{B}_s &\xrightarrow{\mu} \mathbf{B}_s\langle -1\rangle, & \mu(1 \otimes f \otimes 1) &= \partial(f) \otimes 1, \\ \mathbf{B}_s\langle 1\rangle &\xrightarrow{\delta} \mathbf{B}_s \otimes_R \mathbf{B}_s, & \delta(1 \otimes 1) &= 1 \otimes 1 \otimes 1. \end{aligned}$$

Similarly, let m be the R-bimodule morphism

$$\mathbf{B}_s \otimes_R \mathbf{B}_s \xrightarrow{m} \mathbf{B}_s \otimes_R \mathbf{B}_s\langle 2\rangle, \qquad m(1 \otimes f \otimes 1) = \tfrac{1}{2}(1 \otimes \alpha f \otimes 1).$$

(3) Show that the morphisms of R-bimodules

$$\delta \circ \mathrm{pr}_1 + m \circ \delta \circ \mathrm{pr}_2 : \mathbf{B}_s\langle 1\rangle \oplus \mathbf{B}_s\langle -1\rangle \leftrightarrows \mathbf{B}_s \otimes_R \mathbf{B}_s : (\mu \circ m, \mu)$$

are two-sided inverses of each other. *Hint:* For one direction, check that $\mu \circ m \circ \delta = \mathrm{id}$. For the other, rewrite both sides concretely.

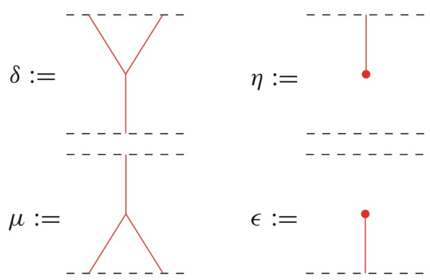
13.2.

Let ϵ, η be the R-bimodule morphisms determined as follows:

$$\mathbf{B}_s \xrightarrow{\epsilon} \mathbf{B}_e\langle 1\rangle, \qquad \epsilon(1\otimes 1) = 1,$$
$$\mathbf{B}_e\langle -1\rangle \xrightarrow{\eta} \mathbf{B}_s, \qquad \eta(1) = \tfrac{1}{2}(\alpha\otimes 1 + 1\otimes\alpha).$$

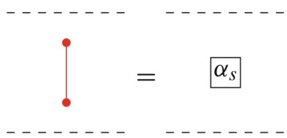
Together, $\epsilon, \eta, \mu, \delta$ endow $\mathbf{B}_s$ with the structure of a Frobenius algebra object in the category of graded R-bimodules. There is an established formalism of *diagrammatics* for such objects.

Problem 13.4. Consider the diagrams below:



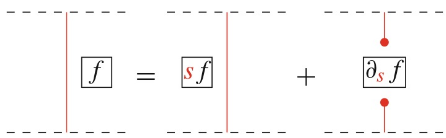
Using these diagrams together with the previous problems, (try to) interpret the following diagrammatic identities:

(1)

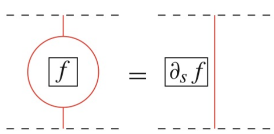


(2)

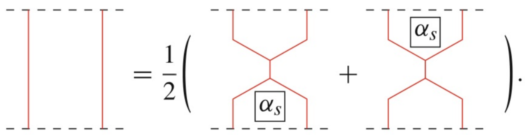
(3)



(4)



(5)



(Of course, these diagrams are stolen from somewhere in print.)

13.3.

From Problem 13.3 and Rose's theorem, we deduce that

$$[\mathsf{K}^b(\mathsf{C}_W)]_\Delta \simeq H_W(\mathsf{x}) \tag{13.1}$$

(at least for $W = \{e, s\}$). The multiplication on the left-hand side is again induced by $\otimes_R$, now extended to a monoidal product on the homotopy category via additivity of degree.

Problem 13.5. Let $\mathcal{R}_s^+$ and $\mathcal{R}_s^-$ be (the homotopy classes of) the complexes

$$\underline{\mathbf{B}_s} \xrightarrow{\epsilon} \mathbf{B}_e\langle 1\rangle \quad \text{and} \quad \mathbf{B}_e\langle -1\rangle \xrightarrow{\eta} \underline{\mathbf{B}_s},$$

where the underlining indicates the terms in degree zero.

(1) Show that $\mathcal{R}_s^+ \otimes_R \mathcal{R}_s^-$ is homotopy equivalent to the complex consisting of $\mathbf{B}_e$ in degree zero.
(2) Under (13.1), what are the images of $[\mathcal{R}_s^+]$ and $[\mathcal{R}_s^-]$ in $H_W(\mathsf{x})$?

13.4.

We now work with $G = \mathrm{SL}_3$ and $W = \langle s, t \mid s^2 = t^2 = (st)^3 = e\rangle$. Following the analogue of the recipe we used for SL_2, we set

$$R := \mathrm{H}^*([pt/(\mathbf{C}^\times)^2], K) = K[\alpha_s, \alpha_t], \quad \text{where } \deg \alpha_s = \deg \alpha_t = 2$$

and let W act on R as follows:

$$\begin{aligned} s \cdot \alpha_s &= -\alpha_s, & s \cdot \alpha_t &= \alpha_s + \alpha_t, \\ t \cdot \alpha_s &= \alpha_s + \alpha_t, & t \cdot \alpha_t &= -\alpha_t. \end{aligned}$$

(This action can be regarded as the W-action on the root system of G.)

Let $\partial_s, \mathbf{B}_e, \mathbf{B}_s$ be defined exactly like $\partial, \mathbf{B}_e, \mathbf{B}_s$ in the SL_2 setting, but using the new definition of R, and using α_s in place of α. Let $\partial_t, \mathbf{B}_t$ be defined in analogy with $\partial_s, \mathbf{B}_s$. Finally, let

$$\mathbf{B}_{sts} = \mathbf{B}_{tst} = R \otimes_{R^W} R\langle 3\rangle.$$

Problem 13.6. (1) In the Hecke algebra $H_W(\mathsf{x})$, let $c_s = \sigma_s + \mathsf{x}^{-1}$ and $c_t = \sigma_t + \mathsf{x}^{-1}$. Verify that

$$c_s c_t c_s - c_s = c_t c_s c_t - c_t.$$

In fact, both sides equal the Kazhdan–Lusztig basis element $c_{sts} = c_{tst}$.

(2) Show that the maps

$$\begin{aligned} \mathbf{B}_s &\to \mathbf{B}_s \otimes_R \mathbf{B}_t \otimes_R \mathbf{B}_s \quad 1 \otimes 1 \mapsto \tfrac{1}{2}(1 \otimes \alpha_t \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha_t \otimes 1), \\ \mathbf{B}_{sts} &\to \mathbf{B}_s \otimes_R \mathbf{B}_t \otimes_R \mathbf{B}_s, \quad 1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \otimes 1 \end{aligned}$$

are injective and that their images jointly span $\mathbf{B}_s \otimes_R \mathbf{B}_t \otimes_R \mathbf{B}_s$.

(3) *Harder:* Show that the maps in (2) induce an isomorphism of graded R-bimodules

$$\mathbf{B}_s \otimes_R \mathbf{B}_t \otimes_R \mathbf{B}_s \simeq \mathbf{B}_{sts} \oplus \mathbf{B}_s.$$

In fact, if we set $\mathbf{B}_{st} = \mathbf{B}_s \otimes_R \mathbf{B}_t$ and $\mathbf{B}_{ts} = \mathbf{B}_t \otimes_R \mathbf{B}_s$, then the graded additive category C_W generated by the $\mathbf{B}_w$ for $w \in W$ under direct sums and grading shifts is closed under $\otimes_R$. There is an isomorphism $[\mathsf{C}_W]_\oplus \xrightarrow{\sim} H_W(\mathsf{x})$, extending the one from the SL_2 case, that sends $\mathbf{B}_{sts} \mapsto c_{sts}$.