10.

Returning to symmetric groups, we present, in this lecture and the next one, two different constructions of a remarkable quotient of the Iwahori Hecke algebra of S_n, called the Temperley–Lieb algebra. The construction today will use a double-centralizer phenomenon called quantum Schur–Weyl duality.

10.1.

We start by reviewing classical Schur–Weyl duality, starting from the double centralizer theorem.

Theorem 10.1 (Double Centralizer)**.** *Let k be any field, E a finite-dimensional k-vector space, and $R \subseteq \mathrm{End}_k(E)$ a semisimple k-subalgebra split over k. Let $R' = \mathrm{End}_R(E)$. Then:*

(1) R' is semisimple.

(2) We have an isomorphism of (R, R')-bimodules

$$E \simeq \bigoplus_i E_i \otimes E'_i,$$

where the sum runs over all isomorphism classes of simple R-modules E_i, and at the same time, $\{V'_i\}_i$ is the set of all isomorphism classes of simple R'-modules.

If, moreover, R' is split over k, then $R \simeq \mathrm{End}_{R'}(V)$.

Proof. Since R is semisimple and split over k, we at least have

$$R \simeq \prod_i \mathrm{End}_k(E_i) \quad \text{as } k\text{-algebras},$$
$$E \simeq \bigoplus_i E_i \otimes \mathrm{Hom}_R(E_i, E) \quad \text{as } R\text{-modules},$$

(In each tensor product above, R only acts on the left factor.) Let $E'_i = \mathrm{Hom}_R(E_i, E)$. Schur's lemma tells us that $\mathrm{End}_R(E_i)$ is a simple k-algebra for all i and $\mathrm{Hom}_R(E_j, E_i) = 0$ for all $j \neq i$. The first fact implies that $\mathrm{End}_R(E_i)$ is simple; the second implies that

$$R' \simeq \prod_i \mathrm{End}_R(E_i) \otimes \mathrm{End}_k(E'_i) \quad \text{as } k\text{-algebras}.$$

So R' is semisimple. If R' is split over k, then we further have $\mathrm{End}_{R'}(E'_i) = k$ for all i. Again, Schur tells us that $\mathrm{Hom}_{R'}(E'_j, E'_i) = 0$ for all $j \neq i$, so we conclude that $\mathrm{End}_{R'}(E) \simeq \prod_i \mathrm{End}_k(E_i) \simeq R$. □

Fix a field k and a finite-dimensional vector space V over k. Let

$$\mathrm{U}_V = \mathrm{U}(\mathfrak{sl}(V)),$$

the universal enveloping algebra of $\mathfrak{sl}(V)$. Via the comultiplication map $\Delta = \Delta^{(2)} : \mathrm{U}_V \to \mathrm{U}_V \otimes \mathrm{U}_V$ given by

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi,$$

the category of finite-dimensional U_V-modules is endowed with a tensor product. In particular, U_V acts diagonally through $\Delta^{(n)}$ on any tensor power of V.

Fix an integer $n \geq 1$. Then there are commuting actions

$$\mathrm{U}_V \curvearrowright V^{\otimes n} \curvearrowleft kS_n,$$

where S_n acts by permuting the copies of V in the tensor product. The k-algebra $\mathrm{End}_{kS_n}(V^{\otimes n})$ is called the *Schur algebra* and denoted $S_{V,n}$.

Theorem 10.2. *If n is invertible in k, then the map $\mathrm{U}_V \to S_{V,n}$ is surjective.*

Proof. First, recall that $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus k$, where the second factor represents scalar elements $c \in \mathfrak{gl}(V)$. The action of $\Delta^{(n)}(c)$ on $V^{\otimes n}$ is given by nc. So if n is invertible in k, then $\mathrm{U}_V = \mathrm{U}(\mathfrak{sl}(V))$ has the same image as $\mathrm{U}(\mathfrak{gl}(V))$ in the Schur algebra.

Next, we observe that $S_{V,n} \simeq \mathrm{Sym}^n(\mathfrak{gl}(V))$. So we want to show that if A is a finite-dimensional k-algebra, then $\mathrm{Sym}^n(A)$ is generated by the image of $\Delta^{(n)} : A \to \mathrm{Sym}^n(A)$. This is an exercise in symmetric function theory: See part (ii) of Lemma 5.18.3 in Etingof *et al.*'s book on representation theory. $\square$

At the same time, recall that if $n!$ is invertible in k, then kS_n is semisimple and split over k. Applying the double centralizer theorem to $E = V^{\otimes n}$ and $R = kS_n$, we deduce:

Corollary 10.3 (Schur–Weyl)**.** *If $n!$ is invertible in k, then $S_{V,n}$ is semsimple and there is a bijection between:*

(1) Irreducible characters $\chi \in \mathrm{Irr}(S_n)$ such that

$$(\chi, \chi_{V^{\otimes n}})_{S_n} \neq 0.$$

(2) Isomorphism classes of simple U_V-modules M such that

$$\mathrm{Hom}_{\mathrm{U}_V}(M, V^{\otimes n}) \neq 0.$$

Moreover, if $S_{V,n}$ is split over k, then the map $kS_n \to \mathrm{End}_{S_{V,n}}(V^{\otimes n})$ is surjective.

10.2.

It turns out that if $n > \dim(V)$, then the map $kS_N \to \mathrm{End}_{S_{V,n}}(V^{\otimes n})$ is not injective. We now focus on the case where $V = k^2$, where we can illustrate this fact more easily.

Recall from the representation theory of $\mathfrak{sl}_2$ that $\mathrm{Sym}^\ell(V)$ is a simple $\mathfrak{sl}_2$-module for all $\ell \geq 0$, and that the decomposition of $V^{\otimes n}$ into these modules (*i.e.*, its Clebsch–Gordan numbers) can be determined using weights. Explicitly, if we write $[\ell]_\mathsf{x} = \mathsf{x}^{\ell-1} + \mathsf{x}^{\ell-3} + \cdots + \mathsf{x}^{1-\ell}$, and define integers $c_{n,\ell}$ by

$$[2]_\mathsf{x}^n = \sum_{\ell \geq 0} c_{n,\ell}[\ell+1]_\mathsf{x},$$

then $c_{n,\ell} = \dim \mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{Sym}^\ell(V), V^{\otimes n})$. So the (U_V, kS_n)-bimodule decomposition of $V^{\otimes n}$ must take the form

$$V^{\otimes n} \simeq \bigoplus_{\ell \geq 0} \mathrm{Sym}^\ell(V) \otimes V_{n,\ell},$$

where, for all ℓ, the kS_n-module $V_{n,\ell}$ is either zero or simple of dimension $c_{n,\ell}$, and the nonzero modules are pairwise non-isomorphic. But the definition of the $c_{n,\ell}$ shows that they vanish unless $n - \ell$ is nonnegative and even. Therefore, only $\lfloor n/2 \rfloor + 1$ of the irreducible characters of S_n can appear, whereas $|\mathrm{Irr}(S_n)|$ is the number of partitions of n, which grows much faster with n.

When $n - \ell$ is nonnegative and even, it turns out that $V_{n,\ell}$ is the irreducible representation of S_n indexed by the two-row partition $(n - \lfloor \ell/2 \rfloor, \lfloor \ell/2 \rfloor) \vdash n$. For instance, if $\ell \in \{0, 1\}$, then this partition is the trivial partition.

10.3.

We continue to keep $V = k^2$. Then $\mathrm{U}_V = k \otimes \mathrm{U}(\mathfrak{sl}_2)$, where

$$\mathrm{U}(\mathfrak{sl}_2) = \frac{\mathbf{Z}[h, e, f]}{\langle [e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f \rangle},$$

the integral form of the universal enveloping algebra of $\mathfrak{sl}_2$. We can take the elements h, e, f to represent, say,

$$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Following Drinfeld–Jimbo, we define (the split form of) the *quantized universal enveloping algebra* of $\mathfrak{sl}_2$ to be the $\mathbf{Z}[\mathsf{x}^{\pm 1}, \frac{1}{\mathsf{x}-\mathsf{x}^{-1}}]$-algebra

$$\mathrm{U}_\mathsf{x}(\mathfrak{sl}_2) = \frac{\mathbf{Z}[\mathsf{x}^{\pm 1}, \frac{1}{\mathsf{x}-\mathsf{x}^{-1}}][K^{\pm 1}, E, F]}{\left\langle [E, F] = \frac{1}{\mathsf{x} - \mathsf{x}^{-1}}(K - K^{-1}),\ KE = \mathsf{x}^2 EK,\ KF = \mathsf{x}^{-2} FK \right\rangle}.$$

Note that the base ring above looks suspiciously similar to the target ring for the HOMFLYPT link invariant. In particular, we cannot specialize this algebra to a $\mathsf{x} \to 1$ limit directly.

Nonetheless, $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)$ participates in a "quantization" of Schur–Weyl duality, where the Iwahori–Hecke algebra replaces kS_n. To simplify the discussion that follows, we set $k = \mathbf{Q}(\mathsf{x})$ and

$$\mathrm{U}_{V,\mathsf{x}} = k \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}, \frac{1}{\mathsf{x}-\mathsf{x}^{-1}}]} \mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2).$$

Then the following formulas define a $\mathrm{U}_{V,\mathsf{x}}$-action on $V = k^2$:

$$E \mapsto \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \quad F \mapsto \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}, \quad K \mapsto \begin{pmatrix} \mathsf{x} & \\ & \mathsf{x}^{-1} \end{pmatrix}.$$

Remarkably, the category of $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)$-modules free of finite rank over k is again endowed with a tensor product. It is induced by a deformation of the coproduct on $\mathrm{U}(\mathfrak{sl}_2)$:

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, \\ \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}. \end{aligned}$$

Note that since $E, F, K^{\pm 1}$ are already constrained by certain relations, it takes work to check that this coproduct is well-defined, let alone co-associative.

In what follows, let $kH_n = k \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}]} H_n$, where H_n is the Iwahori–Hecke algebra for S_n over $\mathbf{Z}[\mathsf{x}^{\pm 1}]$. Recall that kH_n is semisimple and split over k.

Theorem 10.4 (Quantum Schur–Weyl)**.** *For $k = \mathbf{Q}(\mathsf{x})$, there is a kH_n-action on $V^{\otimes n}$ commuting with the $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)$-action:*

$$\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2) \curvearrowright V^{\otimes n} \curvearrowleft kH_n.$$

Moreover, the maps $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2) \to \mathrm{End}_{kH_n}(V^{\otimes n})$ and $kH_n \to \mathrm{End}_{\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)}(V^{\otimes n})$ are surjective.

Sometimes $\mathrm{End}_{kH_n}(V^{\otimes n})$ is called the *quantized Schur algebra*. The kH_n-action on $V^{\otimes n}$ can be constructed explicitly from a deformation of the kS_n-action. Namely, let $\check{\mathcal{R}} : V \otimes V$ be the linear operator given in column notation by

$$\check{\mathcal{R}} = \begin{pmatrix} \mathsf{x} & & & \\ & \mathsf{x}-\mathsf{x}^{-1} & 1 & \\ & 1 & & \\ & & & \mathsf{x} \end{pmatrix}.$$

For $1 \leq i < j \leq n$, let $\check{\mathcal{R}}^{(i,j)} : V^{\otimes n} \to V^{\otimes n}$ be defined through $\check{\mathcal{R}}$ on the ith and jth factors of the tensor product, and by the identity on all other factors. It turns out that $\check{\mathcal{R}}^{(i,j)}$ commutes with $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)$. The action of kH_n on $V^{\otimes n}$ sends $\sigma_i^{-1} \mapsto \check{\mathcal{R}}^{(i,i+1)}$.

The key point is to check that these operators $\check{\mathcal{R}}^{(i,i+1)}$ satisfy the braid relations

$$\check{\mathcal{R}}^{(1,2)}\check{\mathcal{R}}^{(2,3)}\check{\mathcal{R}}^{(1,2)} = \check{\mathcal{R}}^{(2,3)}\check{\mathcal{R}}^{(1,2)}\check{\mathcal{R}}^{(2,3)}.$$

This identity was discovered by way of its relative, the *Yang–Baxeter equation*

$$\mathcal{R}^{(1,2)}\mathcal{R}^{(1,3)}\mathcal{R}^{(2,3)} = \mathcal{R}^{(2,3)}\mathcal{R}^{(1,3)}\mathcal{R}^{(1,2)}.$$

To go between the two, set $\check{\mathcal{R}} = \mathsf{s} \circ \mathcal{R}$, where $\mathsf{s}(v \otimes v') = v' \otimes v$.

Solutions to the Yang–Baxter equations are called *R-matrices*, and appear in the theory of quantum integrable systems. Somewhat confusingly, the R-matrix above can also be written in terms of the action on $V^{\otimes 2}$ of an element of $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2) \hat{\otimes} \mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_2)$ called the *universal R-matrix*.[1]

10.4.

Recall that if q is a prime power, then Iwahori's theorem interprets $H_n(q) := H_n/(\mathsf{x} - q^{1/2})$ as the algebra of $\mathrm{GL}_n(\mathbf{F}_q)$-invariant functions on $\mathcal{B}(\mathbf{F}_q) \times \mathcal{B}(\mathbf{F}_q)$ under a suitable convolution, where $\mathcal{B}(\mathbf{F}_q)$ is the set of complete flags in $\mathbf{F}_q^n$.

The kH_n-module $V^{\otimes n}$ and the algebra $\mathrm{U}_{\mathsf{x}}(\mathsf{sl}_2)$ have similar geometric interpretations. Observe that the kH_n-action on $V^{\otimes n}$ restricts to a H_n-action on $\mathcal{V}^{\otimes n}$, where $\mathcal{V} = \mathbf{Z}[\mathsf{x}^{\pm 1}]^2$. Taking $\mathsf{x} \to q^{1/2}$, the $H_n(q)$-action on $\mathcal{V}^{\otimes n}(q) := \mathbf{Z}[q^{-1/2}] \otimes \mathcal{V}^{\otimes n}$ can be interpreted as a right action by convolution on the $\mathbf{Z}[q^{-1/2}]$-module of functions on

$$\mathcal{G}_n(\mathbf{F}_q) \times \mathcal{B}(\mathbf{F}_q), \quad \text{where } \mathcal{G}_n(\mathbf{F}_q) = \coprod_k \mathcal{G}_{n,k}(\mathbf{F}_q)$$

and $\mathcal{G}_{n,k}$ is the Grassmannian of k-dimensional subspaces of $\mathbf{F}_q^n$ from the zeroth lecture. The quantized Schur algebra admits a similar integral form, whose $\mathsf{x} \to q^{1/2}$ limit can be interpreted in terms of functions on $\mathcal{G}_n(\mathbf{F}_q) \times \mathcal{G}_n(\mathbf{F}_q)$. The action of the latter on $\mathcal{V}^{\otimes n}$ then becomes a left action by convolution.

In fact, quantum Schur–Weyl duality and this geometric interpretation both generalize from $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_n)$ to the infinite family of quantum groups $\mathrm{U}_{\mathsf{x}}(\mathfrak{sl}_n)$. The geometry is due to Beilinson–Lusztig–MacPherson; a more concise version is given in a paper by Grojnowski–Lusztig.

[1] See Losev's note https://gauss.math.yale.edu/~il282/RT/RT13.pdf.

10.5.

We define the *Temperley–Lieb algebra* (over k) to be

$$kTL_n = \mathrm{End}_{\mathrm{U}_x(\mathfrak{sl}_2)}(V^{\otimes n}).$$

Just as the map $kS_n \to \mathrm{End}(V^{\otimes n})$ fails to be injective when $n > \dim(V)$, the map $kH_n \to kTL_n$ fails to be injective when $n > 2$.

Recall that the irreducible representations of S_n appearing in $V^{\otimes n}$ are those indexed by two-row partitions. The bijection between isomorphism classes of (finite-dimensional) simple kS_n-modules and irreducible characters of S_n restricts to a bijection between isomorphism classes of simple kTL_n-modules and irreducible characters of S_n indexed by two-row partitions.

Our notation suggests that kTL_n takes the form $k \otimes_{\mathbf{Z}[x^{\pm 1}]} TL_n$ for some quotient $H_n \to TL_n$. This is indeed the case, as we will discuss next time.