14.

Today we introduce the equivariant constructible derived category as a black-box formalism, like we did with étale cohomology, then review perverse sheaves in this setting. To conclude, we survey what is still missing for us to categorify the Iwahori–Hecke algebra.

Besides Achar, references for the constructible derived category might include the Wikipedia page on "Six operations" and the Romanov–Williamson lecture notes. For perverse sheaves, I recommend Williamson's text, "An Illustrated Guide…" in addition to the book by Beilinson–Bernstein–Deligne–Gabber.

14.1.

In broad terms, a triangulated category D is a framework that lets us apply, to purely algebraic objects, analogues of certain homotopy-theoretic operations: mapping cones and suspensions. These operations ultimately allow us to measure the algebraic objects, through operations known as (co)homology. However, there may be several inequivalent ways of measuring in D. Very roughly, this is similar to how a vector space can support many different coordinate systems at once. The analogues of coordinate systems for triangulated categories are called t-structures.

A *t-structure* on D is a pair of full subcategories $\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0} \subseteq \mathsf{D}$ such that $\mathsf{D}^{\leq i} := \mathsf{D}^{\leq 0}[-i]$ and $\mathsf{D}^{\geq i} := \mathsf{D}^{\geq 0}[-i]$ satisfy:

(1) $\mathsf{D}^{\leq i-1} \subseteq \mathsf{D}^{\leq i}$ and $\mathsf{D}^{\geq i+1} \subseteq \mathsf{D}^{\geq i}$.
(2) $\mathrm{Hom}(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 1}) = 0$.
(3) For all K, there is an exact triangle

$$K' \to K \to K'' \to K'[1] \quad \text{where } K' \in \mathsf{D}^{\leq 0} \text{ and } K'' \in \mathsf{D}^{\geq 1}.$$

A triangulated functor $\mathsf{D} \to \mathsf{D}'$ is *left*, *resp.* *right t-exact* with respect to t-structures on D and D' if and only if it restricts to a functor $\mathsf{D}^{\geq 0} \to (\mathsf{D}')^{\geq 0}$, *resp.* a functor $\mathsf{D}^{\leq 0} \to (\mathsf{D}')^{\leq 0}$. When both are true, we say that it is *t-exact*.

Purely formally, one checks that the inclusions $\mathsf{D}^{\leq i} \subseteq \mathsf{D}$ and $\mathsf{D}^{\geq i} \subseteq \mathsf{D}$ are respectively left and right adjoint to additive *truncation functors* $\tau^{\leq i} : \mathsf{D} \to \mathsf{D}^{\leq i}$ and $\tau^{\geq i} : \mathsf{D} \to \mathsf{D}^{\geq i}$. The *$i$th cohomology functor* defined by the t-structure is

$$\mathcal{H}^i = \tau^{\leq i}\tau^{\geq i}[i] = \tau^{\geq i}\tau^{\leq i}[i] : \mathsf{D} \to \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}.$$

Thus exact triangles in D give rise to long exact sequences on cohomology. A more difficult, but still purely formal, result is:

Theorem 14.1 (Beilinson–Bernstein–Deligne–Gabber)**.** *$\mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0} \subseteq \mathsf{D}$ is always an abelian category, called the* heart *of the t-structure.*

Notably, the proof uses the octahedral axiom. I like the exposition in Section 8 of Murayama's notes from a course taught by Bhargav Bhatt at UMichigan.

Suppose that A is an abelian category and D is the derived category of complexes of objects in A. Then there is a *standard t-structure* on D in which $\mathcal{H}^i$ is the usual ith cohomology functor, and $\mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$ is equivalent to A.

Unfortunately, the converse is not necessarily true: If A is the heart of a t-structure on D, then D need not be the derived category of complexes in A.

14.2.

Let k be an algebraically closed field and ℓ a prime invertible in k. Let X be a scheme of finite type over k. In earlier notes, we sketched a description of the abelian category of constructible $\bar{\mathbf{Q}}_\ell$-sheaves $\mathsf{Shv}(X) = \mathsf{Shv}(X, \bar{\mathbf{Q}}_\ell)$. In particular, we mentioned that the traditional construction of this category is very roundabout: One actually constructs an appropriate triangulated category first, then recovers $\mathsf{Shv}(X)$ as the heart of an appropriate t-structure.

Suppose that G is a smooth algebraic group over k that acts on X. Here there is a generalization of $\mathsf{Shv}(X)$, called the category of *equivariant constructible $\bar{\mathbf{Q}}_\ell$-sheaves* and denoted $\mathsf{Shv}_G(X)$, that roughly corresponds to replacing X with the stack quotient $[X/G]$.[1] Again, it is best defined as the heart of some t-structure on some triangulated category, which we will call the *equivariant constructible derived category* and denote by $\mathsf{D}_G(X) = \mathsf{D}_G(X, \bar{\mathbf{Q}}_\ell)$. In what follows, we simply list the properties of the functors

$$\mathcal{H}^i = \mathcal{H}^0 \circ [i] : \mathsf{D}_G(X) \to \mathsf{Shv}_G(X).$$

Of course, $\mathcal{H}^i(K)$ is called the *ith cohomology sheaf* of K.

There are now several approaches to constructing $\mathsf{D}_G(X)$, surveyed in a doctoral thesis by Vooys at the University of Calgary. The most well-known treatment is due to Bernstein–Lunts, though much of their book works in a topological, not algebro-geometric setting.

14.2.1.

Suppose that Y is another scheme of finite type with a G-action over k, and that $p : Y \to X$ is a G-equivariant morphism over k. Then p induces *derived*

$$\textit{pullbacks } p^*, p^! : \mathsf{D}_G(X) \to \mathsf{D}_G(Y),$$
$$\textit{pushforwards } p_*, p_! : \mathsf{D}_G(Y) \to \mathsf{D}_G(X).$$

These are often written with symbols L, R, which we will omit for convenience, to indicate that they are derived. Some important special cases:

[1] My understanding is that $\mathsf{Shv}_G(X)$ should be a full subcategory of $\mathsf{Shv}([X/G])$, where Shv is defined on the stack side using the lisse-étale topology, following papers of Laszlo–Olsson. Achar takes a different approach where the categories are equivalent by fiat: See §6.8 in his text.

- If p is proper, then $p_! = p_*$.
- If p is smooth of relative dimension d, then $p^! = p^*[2d]$. A useful principle to keep in mind is that *smooth pullbacks* commute with all other sheaf operations (Achar Principle 2.2.11).

Like in classical sheaf theory, *cf.* Hartshorne II.1.18, we have an adjunction

$$\mathrm{Hom}(p^*K, L) \simeq \mathrm{Hom}(K, p_*L)$$

functorial in K, L.

Recall that we set $pt = \mathrm{Spec}\, k$. We always endow pt with the trivial G-action. There is a constant object $\bar{\mathbf{Q}}_\ell \in \mathsf{Shv}_G(pt)$, which represents the functors

$$\mathrm{H}^i := \mathcal{H}^i : \mathsf{D}_G(pt) \to \mathsf{Shv}_G(pt)$$

in the sense that $\mathcal{H}^i(K) = \mathrm{Hom}(\bar{\mathbf{Q}}_\ell, K[i])$ for all i and $K \in \mathsf{D}_G(pt)$.

In terms of the unique map $a : X \to pt$, the *constant sheaf* on X is $(\bar{\mathbf{Q}}_\ell)_X = a^*\bar{\mathbf{Q}}_\ell \in \mathsf{D}_G(X)$, like before. The *dualizing complex* on X is $\omega_X = a^!\bar{\mathbf{Q}}_\ell \in \mathsf{D}_G(X)$.

14.3.

We regard p_* and $p_!$ as relative versions of ordinary and compactly-supported G-equivariant singular cohomology, respectively, due to the following facts.

(1) If $G = \{1\}$, then

$$\mathrm{H}^i((\bar{\mathbf{Q}}_\ell)_X) = \mathrm{H}^i(X), \qquad \mathrm{H}^i(a_!(\bar{\mathbf{Q}}_\ell)_X) = \mathrm{H}^i_c(X),$$

where the right-hand sides denote the ordinary and compactly-supported étale cohomology of X, respectively.

(2) If $k = \mathbf{C}$ and G is arbitrary, then

$$\mathrm{H}^i(a_*(\bar{\mathbf{Q}}_\ell)_X) = \mathrm{H}^i_G(X^{an}), \qquad \mathrm{H}^i(a_!(\bar{\mathbf{Q}}_\ell)_X) = \mathrm{H}^i_{c,G}(X^{an}),$$
$$\mathrm{H}^i(a_*\omega_X) = \mathrm{H}^{\mathrm{BM},G}_{-i}(X^{an}), \qquad \mathrm{H}^i(a_!\omega_X) = \mathrm{H}^G_{-i}(X^{an}),$$

where the right-hand sides denote the G-equivariant ordinary (singular) cohomology, compactly-supported cohomology, Borel–Moore homology, and ordinary homology of the underlying complex-analytic space X^{an}, respectively. (This turns out to be a backwards way of defining homology in terms of cohomology via a generalized Poincaré duality.)

(3) For any $\bar{x} : pt \to X$ and $K \in \mathsf{D}_G(X)$, let $K_{\bar{x}} = \bar{x}^*K$, the *stalk* of K at $\bar{x}$. Then we have

$$\mathrm{H}^i(K_{\bar{x}}) = \mathcal{H}^i(K)_{\bar{x}},$$

(This ultimately reflects the fact that in classical sheaf theory, taking stalks is exact, *cf.* Hartshorne II.1.2.)

For a general object $K \in \mathsf{Shv}(X)$, we define the *ith equivariant ordinary hypercohomology* $\mathrm{H}^i_G(X, K)$ and *compactly-supported hypercohomology* $\mathrm{H}^i_{c,G}(X, K)$ according to

$$\mathrm{H}^i_G(X, K) = \mathrm{H}^i(a_* K), \qquad \mathrm{H}^i_{c,G}(X, K) = \mathrm{H}^i(a_! K).$$

When $K = (\bar{\mathbf{Q}}_\ell)_X$, we write $\mathrm{H}^i_G(X, \bar{\mathbf{Q}}_\ell), \mathrm{H}^i_{c,G}(X, \bar{\mathbf{Q}}_\ell)$ instead.

Theorem 14.2 (Base Change)**.** *Suppose that $p : Y \to X$ is separated and we have a* cartesian *square:*

$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{\pi}} & Y \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{\pi} & X \end{array}$$

Then there are base change equivalences $\pi^* p_! \simeq \tilde{p}_! \tilde{\pi}^*$ and $\pi^! p_* \simeq \tilde{p}_* \tilde{\pi}^!$. They are functorial in p and π.

Corollary 14.3. *Taking $X' = \{\bar{x}\}$ for some G-stable $\bar{x} \in X(k)$, we have*

$$\mathrm{H}^i_{c,G}(Y_{\bar{x}}) \simeq \mathrm{H}^i(\tilde{p}_! \tilde{\pi}^* (\bar{\mathbf{Q}}_\ell)_Y) \simeq \mathrm{H}^i(\pi^* p_! (\bar{\mathbf{Q}}_\ell)_Y) \simeq \mathcal{H}^i(p_! (\bar{\mathbf{Q}}_\ell)_Y)_{\bar{x}}.$$

That is, the cohomology of the (geometric) fibers of p can be computed via the stalks of certain cohomology sheaves.

14.3.1.

In addition to the pullbacks and pushforwards, equivariant constructible derived categories admit a *derived tensor product* $\otimes = \otimes^{\mathrm{L}}$ and *internal Hom* $\mathcal{H}om = \mathrm{R}\mathcal{H}om$ that satisfy:

(1) $\mathcal{H}om((\bar{\mathbf{Q}}_\ell)_X, K) = K$.
(2) $\mathrm{H}^0_G(X, \mathcal{H}om(K, L)) = \mathrm{Hom}(K, L)$.
(3) Local *$\otimes$-$\mathcal{H}om$ adjunction*. $\mathcal{H}om(K \otimes (-), L) \simeq \mathcal{H}om(K, \mathcal{H}om(-, L))$.
(4) Local *Verdier duality*. $\mathcal{H}om(p_! K, L) \simeq p_* \mathcal{H}om(K, p^! L)$.
(5) The local *projection formula*. $p_!(p^* K \otimes L) \simeq K \otimes p_! L$.

Everything above is functorial in K, L.

Remark 14.4. BBDG write $R\underline{\mathrm{Hom}}$ for the internal Hom, whereas Achar writes $R\mathcal{H}om$ for the internal Hom and $R\underline{\mathrm{Hom}}$ for its $*$-pushforward to pt, so the literature has conflicting notation.

By (1)–(2), we have $H^i_G(X, K) = \mathrm{Hom}((\bar{\mathbf{Q}}_\ell)_X, K[i]) = \mathrm{Hom}(\bar{\mathbf{Q}}_\ell, a_* K[i])$. In particular, there is a graded action of $\mathrm{H}^*_G(pt, \bar{\mathbf{Q}}_\ell)$ on $\mathrm{H}^*_G(X, K)$. As a byproduct, $\mathrm{H}^*_G(pt, -) = \mathcal{H}^*$ defines a forgetful functor from $\mathsf{D}_G(pt)$ to the derived category of complexes of finitely-generated $\mathrm{H}^*_G(pt, \bar{\mathbf{Q}}_\ell)$-modules.

Example 14.5. If G is finite (and discrete), then $\mathrm{H}^*_G(pt) = \mathrm{H}^0_G(pt) = \bar{\mathbf{Q}}_\ell G$. In this case, $\mathrm{H}^*_G(pt, -) : \mathsf{D}_G(pt) \to \mathsf{D}(\mathsf{Mod}^{\mathsf{fg}}_{\bar{\mathbf{Q}}_\ell G})$ is an equivalence.

The projection formula implies the *Künneth formula*: Given $K_i \in \mathsf{D}_G(X_i)$ for $i = 1, 2$, we have

$$\mathrm{H}^*_{c,G}(X_1 \times X_2, K_1 \boxtimes K_2) \simeq \mathrm{H}^*_{c,G}(X_1, K_1) \otimes \mathrm{H}^*_{c,G}(X_2, K_2),$$

where $K_1 \boxtimes K_2 = \mathrm{pr}_1^* K_1 \otimes \mathrm{pr}_2^* K_2$. When $k = \mathbf{C}$, and we replace the étale topology with the analytic topology, the analogous Künneth formula with H^*_G in place of $\mathrm{H}^*_{c,G}$ is Theorem 4.3.14 in Dimca's *Sheaves and Topology* and Corollary 2.0.4 in Schürmann's *Topology of Singular Spaces and Constructible Sheaves*. We will assume that this noncompact Künneth formula extends to our étale setting with arbitrary k.

It is useful to set $\mathbf{D} = \mathbf{D}_X := \mathcal{H}om(-, \omega_X)$, so that $\mathbf{D}a_! = a_*\mathbf{D}$ for the map $a : X \to pt$. Note as well that if $G = \{1\}$, then $\mathsf{D}_G(pt) = \mathsf{D}(\mathsf{Vect}_{\bar{\mathbf{Q}}_\ell})$, and under this identification, $\mathbf{D}_{pt}$ sends a complex of vector spaces to its (graded) dual.

Example 14.6. Verdier duality reduces to Poincaré duality when $G = \{1\}$ and X is smooth of dimension d. Indeed, for such X, we have $\omega_X = (\bar{\mathbf{Q}}_\ell)_X[2d]$, from which

$$\mathrm{H}^{-i}(\mathbf{D}_{pt} a_!(\bar{\mathbf{Q}}_\ell)_X) \simeq \mathrm{H}^{-i}(a_*\mathbf{D}_X(\bar{\mathbf{Q}}_\ell)_X) \simeq \mathrm{H}^{-i}(a_*(\bar{\mathbf{Q}}_\ell)_X[2d]).$$

The left-hand side is $\mathrm{H}^i_c(X)^\vee$ and the right-hand side is $\mathrm{H}^{2d-i}(X)$.

14.3.2.

Suppose that we have a closed embedding $i : Z \to X$ and an open embedding $j : V \to X$ of finite-type subschemes of X complementary to each other. Then there is an exact triangle of endofunctors of $\mathsf{D}_G(X)$:

$$j_! j^! \to \mathrm{id} \to i_* i^* \to j_! j^! [1].$$

(Recall that $i_* = i_!$ and $j^! = j^*$.) Here i_* and $j_!$ are called *extension-by-zero* functors.

Example 14.7. Applying this triangle to $(\bar{\mathbf{Q}}_\ell)_X$, then applying $a_!$ for the map $a : X \to pt$, we obtain the usual long exact sequence

$$\cdots \to \mathrm{H}^*_{c,G}(U, \bar{\mathbf{Q}}_\ell) \to \mathrm{H}^*_c(X, \bar{\mathbf{Q}}_\ell) \xrightarrow{i^*} \mathrm{H}^*_c(Z, \bar{\mathbf{Q}}_\ell) \to \mathrm{H}^*_{c,G}(U, \bar{\mathbf{Q}}_\ell)[1] \to \cdots$$

14.3.3.

Lastly, we discuss functors that arise from changing G.

If $H \subseteq G$ is a closed, smooth algebraic subgroup, then it gives rise to a forgetful functor $\mathsf{D}_G(X) \to \mathsf{D}_H(X)$ and two flavors of averaging functor $\mathsf{D}_H(X) \to \mathsf{D}_G(X)$. All of these functors are t-exact with respect to the standard t-structures: See Achar §6.4.3 and Corollary 6.6.3.

If the G-action on X factors through a smooth quotient $G \to G/N =: Q$, then it gives rise to an inflation functor $\mathsf{D}_Q(X) \to \mathsf{D}_G(X)$ and two functors $\mathsf{D}_G(X) \to \mathsf{D}_Q(X)$, the latter corresponding to N-invariants and -coinvariants.

All of these functors are more cleanly understood in terms of morphisms of quotient stacks: namely,

$$p : [X/H] \to [X/G] \quad \text{and} \quad q : [X/G] \to [X/Q].$$

Under this viewpoint, the forgetful functor becomes $p^!$, while the averaging functors become p_*, $p_![2\dim G/H]$; the inflation functor becomes q^*, while the invariants and coinvariants become q_* and $q_![-2\dim N]$. Several adjunctions thereby become visible. Compare Achar §6.6, 6.8.

14.4.

Suppose that G is connected reductive and $X = \mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the flag variety of G. As usual, we write W for the Weyl group. Recall that the G-orbits on $\mathcal{B} \times \mathcal{B}$ are indexed by the elements of W. We write $j_w : O_w \to X$ for the inclusion of the orbit indexed by w.

From our earlier discussions of the Hecke algebra $H^{G^F}_{T^F}(1) \simeq H_W(\mathsf{x})|_{\mathsf{x} \to q^{1/2}}$, we are led to expect that the extensions-by-zero $j_{w,!}(\bar{\mathbf{Q}}_\ell)_{O_w}$ are related to the standard basis elements h_w. Actually, there are two natural settings in which we could work with sheaves on X arising from the orbits O_w.

(I) The first is to work with the non-equivariant category $\mathsf{D}(X)$, merely incorporating the G-action on X through the form of the sheaves $j_{w,!}(\bar{\mathbf{Q}}_\ell)_{O_w}$. The advantage of this setting is that taking $k = \bar{\mathbf{F}}_q$ and fixing an $\mathbf{F}_q$-form of G makes the function-sheaf dictionary available.

Let $\mathsf{D}^b(X) \subseteq \mathsf{D}(X)$ be the full subcategory of objects with cohomology in bounded degrees. We extend the dictionary from $\mathsf{Shv}(X)$ to $\mathsf{D}^b(X)$ simply by taking an Euler characteristic:

$$\mathbf{t}_{K,d}(x) = \sum_i (-1)^i \operatorname{tr}(F^d \mid \mathcal{H}^i(K)) \quad \text{for all } x \in X(\mathbf{F}_{q^d})$$

when $F : X \to X$ corresponds to an $\mathbf{F}_q$-structure.

In his exposition, “Perverse Sheaves on Flag Manifolds and Kazhdan–Lusztig Polynomials…”, Riche uses this non-equivariant setup, but instead of taking $k = \bar{\mathbf{F}}_q$, he stays over $\mathbf{C}$ and works in the analytic topology. He introduces an ad-hoc analogue of the function-sheaf dictionary that still works because the sheaves are so nice.

(II) The second is to work with the equivariant category $\mathsf{D}_G(X)$. One advantage of this setting is that for any Borel $B \subseteq G$, there are isomorphisms of quotient stacks

$$[G\backslash(\mathcal{B} \times \mathcal{B})] \simeq [B\backslash\mathcal{B}] \simeq [B\backslash G/B],$$

allowing us to work with other formulations of our category:

$$\mathsf{D}_G(\mathcal{B} \times \mathcal{B}) \simeq \mathsf{D}_B(\mathcal{B}) \simeq \mathsf{D}_{B\times B}(G).$$

The last version, in particular, will endow our sheaves with an action of $\mathrm{H}^*_{B\times B}(pt)$. Using Künneth, one finds that $\mathrm{H}^*_{B\times B}(pt) \simeq R \otimes R^{\mathrm{op}}$ as a graded ring, where $R = \mathrm{H}^*_T(pt)$ for the maximal torus $T = B/[B, B]$. This observation relates the sheaf theory to the graded R-bimodules from the active learning worksheet, *a.k.a.* Soergel bimodules.

Recall the realization of $H^{G^F}_{T^F}(1)$ as a convolution algebra of functions on $\mathcal{B} \times \mathcal{B}$. Taking $k = \bar{\mathbf{F}}_q$ in setting (I), one can check that $*$-pullback of K, *resp.* !-pushforward of K, corresponds to pullback of $\mathbf{t}_{K,d}$, *resp.* pushforward of $\mathbf{t}_{K,d}$ via integration along fibers. This suggests a sheaf-theoretic convolution $*$ corresponding to the function-theoretic one, given by the same formula in both settings above. Namely, writing $\mathrm{pr}_{i,j} : \mathcal{B}^3 \to \mathcal{B}^2$ for the various two-out-of-three projection maps, we should have

$$K * L = \mathrm{pr}_{1,3,!}(\mathrm{pr}^*_{1,2} K \otimes \mathrm{pr}^*_{2,3} L).$$

Note that the maps $\mathrm{pr}_{i,j}$ are all smooth and proper, giving $\mathrm{pr}^*_{i,j} = \mathrm{pr}^!_{i,j}$ and $\mathrm{pr}_{i,j,!} = \mathrm{pr}_{i,j,*}$.

However, we run into a problem: The elements h_w do not just generate the Hecke algebra, they span it over $\mathbf{Z}[q^{\pm 1/2}]$. So we might want some kind of Grothendieck group generated by the classes $[j_{w,!}(\bar{\mathbf{Q}}_\ell)_{O_w}]$ to be *closed* under the convolution above. Unfortunately, this fails to work, because the convolution products do not split into the desired summands. The problem is the pushforward.

14.5.

The solution is to use a different t-structure on $\mathsf{D}(X)$ or $\mathsf{D}_G(X)$ that is better behaved than the standard t-structure. We first explain this for $G = \{1\}$, then generalize to the equivariant setting later.

To motivate the definition, we need the notion of a *lisse* $\mathbf{k}$-sheaf. Recall that when $\mathbf{k}$ is a finite self-injective ring, we have already defined lisse to mean "étale-locally constant of finite type". For $\mathfrak{o}$ the ring of integers in a finite extension of

$\bar{\mathbf{Q}}_\ell$ with maximal ideal $\mathfrak{m}$, we say that a constructible $\mathfrak{o}$-sheaf $\mathcal{F}$ is *lisse* if and only if its image under

$$\mathsf{Shv}(X, \mathfrak{o}) \to \mathsf{D}^{\leq 0}(\mathsf{Shv}(X, \mathfrak{o}/\mathfrak{m}^j)) \xrightarrow{\mathcal{H}^0} \mathsf{Shv}(X, \mathfrak{o}/\mathfrak{m}^j)$$

is lisse for all j. Finally, a constructible $\bar{\mathbf{Q}}_\ell$-sheaf is *lisse* if and only if it arises from some $\mathfrak{o}$ and some lisse $\mathfrak{o}$-sheaf along $\mathsf{Shv}(X, \mathfrak{o}) \to \mathsf{Shv}(X, \bar{\mathbf{Q}}_\ell)$. The main caveat about lisse $\bar{\mathbf{Q}}_\ell$-sheaves is that, unlike lisse $\mathbf{k}$-sheaves for $\mathbf{k}$ finite self-injective, they need not trivialize in an étale open neighborhood of a given geometric point.

For any lisse $\mathbf{k}$-sheaf $\mathcal{L}$ on X, we define its *dual* to be $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathbf{k}_X)$. We notice that if X is smooth of dimension d, so that $\omega_X = \mathbf{k}_X[2d]$, then

$$\mathbf{D}_X(\mathcal{L}[d]) = \mathbf{D}_X(\mathcal{L})[-d] = \mathcal{L}^\vee[d].$$

Above, $\mathcal{L}[d]$ and $\mathcal{L}^\vee[d]$ both occupy degree $-d$ in the standard t-structure, which suggests that we might look for a new t-structure in which these objects both occupy degree 0.

For general X, the *perverse t-structure* (with respect to the "middle perversity") on $\mathsf{D}(X)$ is the pair ${}^p\mathsf{D}(X)^{\leq 0}, {}^p\mathsf{D}(X)^{\geq 0} \subseteq \mathsf{D}(X)$ defined in terms of the standard one by

$$\begin{aligned} {}^p\mathsf{D}(X)^{\leq 0} &= \{K \mid \dim \operatorname{supp}(\mathcal{H}^{-i}(K)) \leq i\}, \\ {}^p\mathsf{D}(X)^{\geq 0} &= \{K \mid \dim \operatorname{supp}(\mathbf{D}\mathcal{H}^{-i}(K)) \leq i\}, \end{aligned}$$

where $\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$. The objects of the heart are called *perverse sheaves*. We write $\mathsf{Perv}(X) = {}^p\mathsf{D}(X)^{\leq 0} \cap {}^p\mathsf{D}(X)^{\geq 0}$, and write ${}^p\mathcal{H}^i$ in place of $\mathcal{H}^i$ to indicate the *ith perverse cohomology sheaf* functor. The following facts about $\mathsf{Perv}(X)$ help us to picture it:

Example 14.8 (Goresky–MacPherson–Deligne)**.** If X is smooth of dimension d, and $\mathcal{L} \in \mathsf{Shv}(X, \bar{\mathbf{Q}}_\ell)$ is lisse, then $\mathcal{L}[d] \in \mathsf{Perv}(X)$.

Theorem 14.9 (BBDG)**.** *For any k-scheme X of finite type,* $\mathsf{Perv}(X)$ *is both artinian and noetherian. Hence every object is of finite length.*

Theorem 14.10 (Beilinson)**.** *For any k-scheme X of finite type, the inclusion* $\mathsf{Perv}(X) \subseteq \mathsf{D}(X)$ *extends to a triangulated equivalence* $\mathsf{D}^b\mathsf{Perv}(X) \xrightarrow{\sim} \mathsf{D}^b(X)$.

Remark 14.11. Beilinson's theorem will not generalize to the G-equivariant setting, even when $G = \mathbf{G}_m$ and $X = pt$. As for the theorem about finite length, I actually do not know a reference for the equivariant generalization.

For any k-scheme Z of finite type and (Zariski) open $j : V \to Z$, there is an *intermediate extension functor* $j_{!*} : \mathsf{Perv}(V) \to \mathsf{Perv}(Z)$ defined by

$$j_{!*}E = \mathrm{im}({}^p\mathcal{H}^0(j_!E) \to {}^p\mathcal{H}^0(j_*E)).$$

It is the unique extension of E to Z with no subquotients supported on $Z \setminus V$.

Theorem 14.12 (Goresky–MacPherson–Deligne, BBDG)**.** *In the setup above, if E is simple, then $j_{!*}E$ is simple.*

On a general k-scheme X of finite type, every simple object of $\mathsf{Perv}(X)$ arises in the form $i_! j_{!*}\mathcal{L}[d]$ for some closed $i : Z \to X$, nonempty open $j : V \to Z$ of dimension d, and simple lisse $\bar{\mathbf{Q}}_\ell$-sheaf $\mathcal{L}$.

In practice, people abuse notation and write $j_{!*}E$ in place of $i_! j_{!*}E$ when X is clear from context. For any *smooth* dense open $V \subseteq Z$, we write

$$IC_Z = j_{!*}(\bar{\mathbf{Q}}_\ell)_V[\dim V],$$

and say that IC_Z is the *intersection cohomology complex*, or *IC complex*, of $Z = \bar{V}$ in X. In particular, the hypercohomology

$$\mathrm{H}^*(X, IC_X)$$

is called the *intersection cohomology* of X.

When $k = \mathbf{C}$, the (contravariant) functor $X \mapsto \mathrm{H}^*(X, IC_X)$ recovers (a cohomological form of) the *intersection homology* theory of Goresky–MacPherson, which preceded and inspired the theory of perverse sheaves.

Theorem 14.13 (BBDG)**.** *If $k = \mathbf{C}$ or $k = \bar{\mathbf{F}}_q$ and $p : Y \to X$ is any* proper *morphism of schemes of finite type over k, then there is a (generally noncanonical) isomorphism*

$$p_*IC_Y = \bigoplus_i {}^p\mathcal{H}^i(p_*IC_Y)[-i]$$

in $\mathsf{D}(X)$. Moreover, ${}^p\mathcal{H}^i(p_*IC_Y)$ is a semisimple object of $\mathsf{Perv}(X)$ for all i.

14.6.

Now suppose that G is an arbitrary smooth algebraic group acting on X.

We want an analogous t-structure ${}^p\mathsf{D}_G(X)^{\leq 0}, {}^p\mathsf{D}_G(X)^{\geq 0} \subseteq \mathsf{D}_G(X)$ such that the heart $\mathsf{Perv}_G(X)$ admits a forgetful functor

$$\mathsf{Perv}_G(X) \to \mathsf{Perv}(X).$$

It turns out that we can just define the equivariant version to be the pullback of ${}^p\mathsf{D}(X)^{\leq 0}, {}^p\mathsf{D}(X)^{\geq 0}$ along the forgetful functor $\mathsf{D}_G(X) \to \mathsf{D}(X)$. This makes the forgetful functor t-exact with respect to the perverse t-structures. See Achar page 287 and Theorem 6.4.10.

Remark 14.14. A sanity check: The definition of ${}^p\mathsf{D}(X)^{\leq 0}$, ${}^p\mathsf{D}(X)^{\geq 0}$ involves dimension conditions on certain subschemes of X. If we tried to imitate this with the stack $[X/G]$ in place of the scheme X, then we would run into problems with negative dimensions. The dimension conditions should really be rewritten as codimension conditions, which don't change when we replace all subschemes involved with their stack quotients by G.

All of the theorems for $\mathsf{Perv}(X)$ that we discussed, except Beilinson's theorem about $\mathsf{D}^b\mathsf{Perv}(X)$, have analogues for $\mathsf{Perv}_G(X)$. The category $\mathsf{Perv}_G(X)$ is described more explicitly in Achar Chapter 6.

14.7.

We return to G connected reductive and $X = \mathcal{B} \times \mathcal{B}$. In either setting (I) or setting (II) from earlier, we write

$$E_w := j_{w,!*}(\bar{\mathbf{Q}}_\ell)_{O_w}[\dim O_w].$$

Above, note that $\dim O_w = \ell(w) + \dim \mathcal{B}$.

We will find that via the decomposition theorem, the additive category generated by the objects $E_w[m]$ for varying $w \in W$ and $m \in \mathbf{Z}$ is closed under convolution. This suggests working with the split Grothendieck group of this category.

The remaining problem, then, is that we still do not have an element of this Grothendieck group that supplies either the variable x or the number $q^{1/2}$. Our earlier discussion of the function-sheaf dictionary suggests the solution: We need input from an $\mathbf{F}_q$-form of G. This leads us to study so-called *mixed* perverse sheaves, whose beautiful structure is worked out in Chapter 5 of BBDG's book.