6.

Today, we discuss a big-picture overview of the consequences of Deligne–Lusztig theory and Lusztig's subsequent work on finite reductive groups. This is stolen from online notes by Chao Li.

6.1.

Suppose that G is a reductive algebraic group with Frobenius F and F-stable maximal torus T. If T is maximally split, *i.e.*, contained in an F-stable Borel $B \subseteq G$, then we can perform *parabolic induction* of representations from T^F to G^F in the spirit of Harish-Chandra: Pull back from T^F to B^F, then induct from B^F to G^F. We said that the resulting representations (and/or their summands) are called the *principal series*. An irreducible character of G^F is *cuspidal* if and only if it does not occur in any principal series.

The key idea of Deligne–Lusztig—which they attribute to Macdonald—is to obtain the other irreducible characters of G^F by constructing, for all other F-stable maximal tori $S \subseteq G$, an analogous induction functor from S^F to G^F. What we have actually presented thus far is how to construct this functor when S^F is explicitly conjugate to T^{wF} for some $w \in W$. For such S and a choice of $g \in G(k)$ such that $S = gTg^{-1}$, the virtual character that we denote by $R_{w,\theta}$ is, in other texts, constructed in terms of $\psi := {}^g\theta$ and denoted $R_{S,\psi}$. By what we proved earlier, any two choices for g differ by an element of G^F, and hence, ψ is determined by θ up to conjugation by G^F.

It turns out that if characters ψ, ψ' of S^F are G^F-conjugate, then $R_{S,\psi} = R_{S,\psi'}$ as class functions on G^F. So above, one could take $R_{w,\theta}$ as a definition of $R_{S,\psi}$. However, it is also possible to construct $R_{S,\psi}$ from S and ψ without reference to the maximally split torus T. We may leave this to a problem set.

6.2.

Since every F-stable maximal torus S satisfies $S^F = T^{wF}$ for some w, the $R_{w,\theta}$ comprise all the virtual characters that we get "geometrically" from such tori. The next problem is to determine which ones contribute the same irreducible summands as each other.

Last time, we stated a formula of Deligne–Lusztig implying that if $w, w' \in W$ belong to different F-conjugacy classes, then $R_{w,\theta}, R_{w'\theta'}$ are orthogonal for every pair of characers θ of T^{wF} and θ' of $T^{w'F}$. But we have also seen an example where w, w' are not F-conjugate and $R_{w,1}, R_{w',1}$ have the same *virtual* summands: For $G = \mathrm{SL}_2$ under the standard Frobenius, $R_{e,1} = 1 + \rho$ and $R_{s,1} = 1 - \rho$ for 1 the trivial character and ρ the Steinberg character.

Deligne–Lusztig found a stricter condition that rules out this sort of situation. As motivation, observe that since $G(k) = \bigcup_{m \geq 1} G^{F^m}$, every pair of maximal tori S, S' is conjugate under G^{F^m} for some $m \geq 1$. Since S is commutative, there is a group homomorphism

$$\mathbf{N}_m : S^{F^m} \to S^F,$$

called the *Galois norm*, that sends any element to the product of its conjugates under $F^0, F^1, \ldots, F^{m-1}$. Fix characters ψ of S^F and ψ' of $(S')^F$. We say that the pairs (S, ψ) and (S', ψ') are *geometrically conjugate* if and only if there exists some $m \geq 1$ and $g \in G^{F^m}$ such that $S' = {}^g S$ and $\psi' \circ \mathbf{N}_m = {}^g(\psi \circ \mathbf{N}_m)$. What follows is Corollary 6.3 in Deligne–Lusztig's paper.

Theorem 6.1 (Deligne–Lusztig)**.** *If $R_{S,\psi}, R_{S',\psi'}$ share an irreducible summand, then (S, ψ) and (S', ψ') are geometrically conjugate.*

6.3.

We also stated a formula about Lefschetz numbers, reducing the calculation of a Lefschetz function on G^F to those of other Lefschetz functions (for smaller schemes) on the subset of unipotent elements. It turns out that this formula reduces the calculation of $R_{S,\psi}$ to the case where $\psi = 1$, at the cost of replacing G with a collection of smaller reductive algebraic groups. For clarity in what follows, we will write $R_S^G(\psi)$ in place of $R_{S,\psi}$.

For any $g \in G^F$, let $g = g_s g_u = g_u g_s$ be its *Jordan decomposition*. This decomposition is uniquely determined by requiring g_s to be *semisimple*, meaning an element of some maximal torus of G, and g_u unipotent. It turns out that $g_s, g_u \in G^F$ as well, and that the centralizer $C(g_s) = C_G(g_s)$ is a reductive algebraic group. Let $C(g_s)^\circ \subseteq C(g_s)$ be the connected component at the identity. Observe that if g_s is contained in a torus, then the torus is contained in $C(g_s)^\circ$.

Theorem 6.2 (Deligne–Lusztig)**.** *For any F-stable maximal torus S and element $g \in G^F$, we have*

$$R_{S^F}^{G^F}(\psi)(g) = \frac{1}{|(C(g_s)^\circ)^F|} \sum_{\substack{x \in G^F \\ g_s \in xS^F x^{-1}}} {}^x\psi(g_s) R_{xS^F x^{-1}}^{(C(g_s)^\circ)^F}(1)(g_u).$$

This formula simplifies substantially when $g = g_s$. In particular, when $g = 1$, Deligne–Lusztig use the formula to express the dimension of any irreducible character ρ of G^F as a linear combination of the multiplicities $(\rho, R_{S^F}^{G^F}(\psi))_{G^F}$, running over all F-stable maximal tori S and characters ψ. From this they deduce their Corollary 7.7:

Corollary 6.3 (Deligne–Lusztig)**.** *Every irreducible representation of G^F occurs as a virtual summand of $R_{S,\psi}$ for some S, ψ: hence, of $R_{w,\theta}$ for some w, θ.*

6.4.

From these results, we begin to glimpse how the Deligne–Lusztig induction functors give structure to the set of irreducible characters of G^F, when G is a connected reductive algebraic group over k with Frobenius F.

(1) First, it is partitioned into subsets indexed by the geometric conjugacy classes of pairs (S, ψ), where S is an F-stable maximal torus of G and ψ is a character of S^F. Let $\mathcal{S}_{G,F}$ denote the set of such classes.
(2) Second, the values of the characters associated with a given (S, ψ) can be computed from analogous values where G is replaced by a smaller reductive group, S is replaced by a conjugate torus, and ψ is replaced by the trivial character.

Remarkably, we can repackage this structure in terms of the geometry of a different group. For any connected reductive G with Frobenius F, there is another connected reductive algebraic group $G^\vee$ over k, and a Frobenius on $G^\vee$ that we again denote by F, with the following properties.

(1) $\mathcal{S}_{G,F}$ is in bijection with the set of semisimple $G^\vee(k)$-conjugacy classes of $(G^\vee)^F$.
(2) In (1), the (single) geometric conjugacy class of pairs $(S, 1)$ corresponds to the conjugacy class of the identity element in $(G^\vee)^F$.
(3) The operation $(G, F) \mapsto (G^\vee, F)$ is involutive.

Granting its existence, we can state Lusztig's classification theorem. For any semisimple element $g \in (G^\vee)^F$, whose $G^\vee(k)$-conjugacy class (g) corresponds to a class $[(S, \psi)] \in \mathcal{S}_{G,F}$, let $\mathcal{E}(G^F, (g))$ denote the set of irreducible characters of G^F with nonzero multiplicity in $R_{S,\psi}$.

Theorem 6.4 (Deligne–Lusztig, Lusztig)**.** *We have a partition*

$$\operatorname{Irr}(G^F) = \coprod_{(g)} \mathcal{E}(G^F, (g)),$$

where (g) runs over the semisimple $G^\vee(k)$-conjugacy classes of $(G^\vee)^F$. Moreover, writing $H_g = C_{G^\vee}(g)^\vee$, so that $H_g^\vee = C_{G^\vee}(g)$, we have bijections

$$\begin{aligned} \mathcal{E}(G^F, (g)) &\xrightarrow{\sim} \mathcal{E}(H_g, (1)), \\ \rho &\mapsto \rho_u, \end{aligned}$$

such that the following property holds: Writing $[(S_g, \psi_g)] \in \mathcal{S}_{H_g,F}$ for the class corresponding to the $H_g^\vee(k)$-conjugacy class of g in $(H_g^\vee)^F$, we have

$$(\rho, R_{S,\psi})_{G^F} = \pm(\rho_u, R_{S_g,\psi_g})_{H_g^F},$$

where the sign can be made explicit and only depends on $G, S, (g)$.

We say that $\mathcal{E}(G^F, (g))$ is the *Lusztig series* indexed by (g). Hence, when (g) corresponds to the geometric conjugacy class of (T, θ) for a maximally split T, the Lusztig series for (g) is the principal series indexed by θ.

6.5.

The construction of $G^\vee$ requires some background from Lie theory: the classification of reductive algebraic groups in terms of root data, due to the work of Killing, É. Cartan, and Chevalley. Namely, we take the root datum of $G^\vee$ to be that dual to the root datum of G. It turns out that by the existence of F-stable Borel pairs, the Frobenius on G is determined by an automorphism of its root datum, and the Frobenius on $G^\vee$ can be defined in terms of an appropriate dual automorphism.

A simpler statement is Deligne–Lusztig Proposition 5.7 below. To state it, let us fix compatible isomorphisms $\mu_m(\bar{\mathbf{Q}}_\ell) \simeq (\frac{1}{m}\mathbf{Z})/\mathbf{Z}$ for all $m \geq 1$, where $\mu_m(\bar{\mathbf{Q}}_\ell)$ is the set of mth roots of unity in $\bar{\mathbf{Q}}_\ell$.

Proposition 6.5 (Deligne–Lusztig)**.** *Let $T \subseteq G$ be a maximally split F-stable maximal torus. Then there is a bijection*

$$\mathcal{S}_{G,F} \xrightarrow{\sim} [(\mathrm{X}(T) \otimes \mathbf{Q}/\mathbf{Z})/W]^F,$$

where $\mathrm{X}(T)$ is the lattice of characters $T \to \mathbf{G}_m$, on which $W \rtimes \langle F \rangle$ acts by precomposition. If F corresponds to an $\mathbf{F}_q$-form of G, then for any torus $S = gTg^{-1}$ and character $\psi = {}^g\theta$, the map sends $[(S, \psi)]$ to the image of θ under

$$\mathrm{Hom}(T^{F^m}, \bar{\mathbf{Q}}_\ell^\times) \simeq \mathrm{X}(T) \otimes (\tfrac{1}{|\mathbf{F}_{q^m}^\times|}\mathbf{Z})/\mathbf{Z}.$$