18.

Today we discuss how to geometrize the cocenters of the Iwahori–Hecke algebras $H_W(\mathsf{x})$ using the so-called horocycle correspondence. In doing so, we introduce the Grothendieck–Springer simultaneous resolution and unipotent character sheaves.

18.1.

Recall a result that we stated some time ago: If $K = K'(\mathsf{x})$, where $K' \supseteq \mathbf{Q}$ is a splitting field for W, and we write $KH_W = K \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}]} H_W(\mathsf{x})$, then there is an isomorphism of K-algebras $KW \simeq KH_W$. In general, it does not take $w \in W$ to the standard element σ_w, or even to the Kazhdan–Lusztig element c_w, but to something stranger. It induces an isomorphism of cocenter maps:

$$\begin{array}{ccc} KW & \overset{\sim}{\longleftrightarrow} & KH_W \\ \downarrow & & \downarrow \\ KW/[KW, KW] & \overset{\sim}{\longleftrightarrow} & KH_W/[KH_W, KH_W] \end{array}$$

Recall that the K-linear dual of $KW/[KW, KW]$ is the space of (K-valued) traces on KW, which is freely spanned by the irreducible characters of W. So the dimension of $KW/[KW, KW]$, which is also that of $KH_W/[KH_W, KH_W]$, equals the number of such characters.

So if we want a geometric interpretation of the cocenter of KH_W, or rather, $H_W(\mathsf{x})$, then we might seek to relate our geometric interpretation of $H_W(\mathsf{x})$ to representations of W.

18.2.

Take $k = \bar{\mathbf{F}}_q$ and $G, F, \mathcal{B}$ as usual: in particular, so that G is connected, smooth, and reductive. We assume that W is the Weyl group of G. Recall that for $G = \mathrm{PGL}_n$, we discussed how pullback along the G-equivariant action map

$$\begin{aligned} G \times \mathcal{B} &\xrightarrow{act} \mathcal{B} \times \mathcal{B}, \\ (g, B) &\mapsto (gBg^{-1}, B) \end{aligned}$$

can be viewed as an analogue of the closure operation $\beta \mapsto \hat{\beta}$ on braids $\beta \in Br_n$. Recall, as well, that we motivated *act* in terms of the simpler *diagonal* map $\mathrm{id} \times \mathrm{id} : O_e = \mathcal{B} \to \mathcal{B} \times \mathcal{B}$. Somewhere in between these options is their fiber product, which we will denote $\tilde{G} \to \mathcal{B} \times \mathcal{B}$. At the level of points,

$$\begin{aligned} \tilde{G} &= \{(g, B) \in G \times \mathcal{B} \mid gBg^{-1} = B\} \\ &= \{(g, B) \in G \times \mathcal{B} \mid g \in B\}. \end{aligned}$$

We can check that under the isomorphism of stacks $[G\backslash(G \times \mathcal{B})] \xrightarrow{\sim} [G/\mathrm{Ad}(B)]$, the substack $[G\backslash\tilde{G}]$ corresponds to $[B/\mathrm{Ad}(B)]$. The scheme $\tilde{G}$ is related to W through the following fact: If $g \in G(k)$ is sufficiently generic, then W acts simply transitively on the set of Borels containing g. That is, the forgetful map

$$\pi : \tilde{G} \to G$$

restricts to a W-cover over some dense open locus. It is called the *Grothendieck alteration* or the *Grothendieck–Springer simultaneous resolution*, for reasons that we explain later.

18.3.

To give more detail, recall some definitions: An element $g \in G(k)$ is *regular* if and only if its centralizer in G has minimal dimension, as an algebraic group, among elements of $G(k)$. It is *semisimple*, *resp.* *unipotent*, if and only if it is mapped to a diagonalizable, *resp.* unipotent, element under any algebraic representation $G \to \mathrm{GL}(V)$ (with V a vector space over k), or equivalently, some faithful algebraic representation of G.

Remark 18.1. We previously used the last two definitions to state the existence and uniqueness of a Jordan decomposition $g = g_s g_u = g_u g_s$ for any g, with g_s semisimple and g_u unipotent. A confusing point here is that if g is semisimple, then $g = g_s$, but if g is unipotent, then it can happen that $g \neq g_u$.

The reason is that Jordan decomposition in $\mathrm{GL}(V)$ is bootstrapped from an *additive* version in $\mathrm{End}(V) \supseteq \mathrm{GL}(V)$. Explicitly, suppose that $g = \xi_s + \xi_n$ in $\mathrm{End}(V)$ with ξ_s diagonalizable and ξ_n *nilpotent* such that $\xi_s\xi_n = \xi_n\xi_s$. If $g \in \mathrm{GL}(V)$, then $\xi_s \in \mathrm{GL}(V)$, so the multiplicative decomposition of g is given by $g_s = \xi_s$ and $g_u = 1 + \xi_s^{-1}\xi_n = 1 + \xi_n\xi_s^{-1}$.

By Milne Exercise 17.3(b), g is regular if and only if the semisimple part g_s in its Jordan decomposition is regular. (Note that Milne initially defines regularity in a different way.)

The above conditions on g can be rewritten in terms of subvarieties of G. To explain how this works for regularity, form the group scheme of centralizers

$$I := \{(x, z) \in G \times G \mid zxz^{-1} = x\} \xrightarrow{\mathrm{pr}_1} G.$$

By the upper semicontinuity of fiber dimension,[1] there is a nonempty open subvariety $G^{\mathrm{reg}} \subseteq G$ such that $\dim I_g$ is minimized among $g \in G(k)$ precisely when $g \in G^{\mathrm{reg}}(k)$. We say that G^{reg} is the *regular locus*. Similarly, let $G^{\mathrm{ss}} \subseteq G$ denote the *semisimple locus*, and let $G^{\mathrm{rs}} = G^{\mathrm{reg}} \cap G^{\mathrm{ss}}$, the *regular semisimple*

[1] See https://mathoverflow.net/q/193

locus. Since regularity and semisimplicity are preserved by conjugation, these loci are all stable under $\mathrm{Ad}(G)$.

By Milne Corollary 17.36, every semisimple element of $G(k)$ belongs to $T(k)$ for some maximal torus $T \subseteq G$. By the conjugacy of maximal tori, we deduce that for any fixed T, the composition $T \to G^{\mathrm{ss}} \to [G^{\mathrm{ss}}/\mathrm{Ad}(G)]$ is surjective on k-points. It restricts to a map $T^{\mathrm{reg}} \to [G^{\mathrm{rs}}/\mathrm{Ad}(G)]$. In fact we have a stronger result, stated in terms of affine GIT quotients $X /\!\!/ H := k[X]^H$:

Theorem 18.2 (Chevalley Restriction)**.** *The maps $T \to G^{\mathrm{ss}} \to G$ descend to isomorphisms of varieties*

$$T /\!\!/ W \xrightarrow{\sim} G^{\mathrm{ss}} /\!\!/ \mathrm{Ad}(G) \xrightarrow{\sim} G /\!\!/ \mathrm{Ad}(G),$$

which further restrict to an isomorphism $T^{\mathrm{reg}} /\!\!/ W \xrightarrow{\sim} G^{\mathrm{rs}} /\!\!/ \mathrm{Ad}(G)$.

In fact, Chevalley worked with the Lie algebras, and only in characteristic zero. The statement at the level of algebraic groups, and in positive characteristic, is proved in §3 of an exposition by Springer–Steinberg titled "Conjugacy Classes", in a volume titled *Seminar on Algebraic Groups and Related Finite Groups.*

We can define a map $\tilde{G} \to T$ as follows. First, we claim that if B, B' are any two Borels of G, then there is a canonical isomorphism between their quotients by their respective derived subgroups U, U'. Indeed, we know that $B' = gBg^{-1}$ and $U' = gUg^{-1}$ for some $g \in G(k)$; we then check that the induced isomorphism $B/U \xrightarrow{\sim} B'/U'$ does not depend on g. Thus we may identify all of these quotients with the same algebraic group T_G over k, which is sometimes called the *universal Cartan torus* of G. There is a map

$$\begin{aligned} \tilde{G} &\to T_G, \\ (g, B) &\mapsto g \mod [B, B]. \end{aligned}$$

Henceforth, we fix a particular T arising from some Borel $B = T \ltimes [B, B]$ and the resulting identification $T = T_G$.

Let $\pi^{\mathrm{rs}} : \tilde{G}^{\mathrm{rs}} \to G^{\mathrm{rs}}$ be the pullback of $\pi : \tilde{G} \to G$. The map $\tilde{G} \to T$ restricts to a map $\tilde{G}^{\mathrm{rs}} \to T^{\mathrm{reg}}$. Henceforth, we write the G-action on $\tilde{G}$ as a right, not left, action, to emphasize the equivariance of π and π^{rs}. We can now state:

Theorem 18.3 ($\approx$ Springer)**.** *The square*

$$\begin{array}{ccc} [\tilde{G}^{\mathrm{rs}}/G] & \longrightarrow & T^{\mathrm{reg}} \\ {\scriptstyle \pi^{\mathrm{rs}}}\downarrow & & \downarrow \\ [G^{\mathrm{rs}}/\mathrm{Ad}(G)] & \longrightarrow & G^{\mathrm{rs}} /\!\!/ G = T^{\mathrm{reg}} /\!\!/ W \end{array}$$

is cartesian. The right-hand vertical arrow is an étale cover with Galois group W, and hence, the same is true of the left-hand vertical arrow.

Example 18.4. Take $G = \mathrm{GL}_n$ and T the diagonal torus. We identify W with S_n. The Chevalley map $[G/\mathrm{Ad}(G)] \to T /\!\!/ W$ corresponds to the map that sends any conjugacy class of G to the unordered multiset of diagonal entries in its Jordan normal form. Conversely, any such multiset determines a unique semisimple conjugacy class: namely, the class of diagonalizable matrices with those eigenvalues. The conjugacy class is regular if and only if the values are pairwise distinct.

Lifting along $T \to T /\!\!/ W$ corresponds to imposing a total ordering on an unordered multiset of eigenvalues. If the values are pairwise distinct, then W acts simply transitively on their total orderings.

Now fix a semisimple element $g \in G(k)$. The set $T_{[g]}$ of elements of $T(k)$ conjugate to g can be identified with the set of total orderings on the eigenvalues of g. If g is also regular, then W acts simply transitively on $T_{[g]}$, and at the same time, the only flags in k^n stabilized by these elements are those split by the coordinate axes: *i.e.*, the W-translates of the standard flag. In this case, fixing an element $t \in T_{[g]}$ determines an equivariant bijection between the elements of $T_{[g]}$ and these flags, or equivalently, the W-conjugates of the upper-triangular Borel B: *i.e.*, the set of Borels containing T.

Writing $g = hth^{-1}$, we conclude that the Borels containing g are precisely those of the form $hwBw^{-1}h^{-1}$. In this way, the fiber of $\pi^{\mathrm{rs}} : \tilde{G}^{\mathrm{rs}} \to G$ above g is the pullback of $T_{[g]}$.

18.4.

All the varieties that we discussed above can be defined over $k_1 = \mathbf{F}_q$, not just over k. Henceforth, we assume that F acts trivially on W. Writing $\tilde{G}_1, G_1^{\mathrm{reg}}, G_1^{\mathrm{ss}}, G_1^{\mathrm{rs}}, T_1^{\mathrm{reg}}$ for the $\mathbf{F}_q$-structures on $\tilde{G}, G^{\mathrm{reg}}, G^{\mathrm{ss}}, G^{\mathrm{rs}}, T^{\mathrm{reg}}$, we find that Theorem 18.3 remains true over k_1, not just over k.

We can further check that W acts freely on T^{reg}, hence on T_1^{reg}. In general, it turns out that if $t \in T(k)$ has stabilizer W_t in W, then its (the identity component of) stabilizer in G is a Levi subgroup of G with Weyl group W_t. For t to be regular, this Levi must be a torus, hence equal to T; in this case, W_t is trivial.

So the map $T_1^{\mathrm{reg}} \to T_1^{\mathrm{reg}} /\!\!/ W$ is an étale cover with deck transformation group W. By Theorem 18.3, the same holds for $\pi_1^{\mathrm{rs}} : \tilde{G}_1^{\mathrm{rs}} \to G_1^{\mathrm{rs}}$. In particular, the constant sheaf $(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1^{\mathrm{rs}}} \in \mathsf{D}_{G_1}(G_1)$ admits a W-equivariant structure, so its pushforward admits a W-action that we can decompose into isotypic summands:

$$\pi^{\mathrm{rs}}_{1,!}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1^{\mathrm{rs}}} = \pi^{\mathrm{rs}}_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1^{\mathrm{rs}}} = \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi \otimes \mathcal{L}_{1,\chi}.$$

Here, $\mathcal{L}_\chi$ is lisse for all χ. The hypothesis that χ is irreducible implies that $\mathcal{L}_{\chi,1}\langle \dim G\rangle$ is simple as an object of $\mathsf{Perv}_{G_1}(G_1)$.

It turns out that G_1^{rs} forms a dense open of G_1^{reg}. (For $G = \mathrm{GL}_n$ under the standard Frobenius, this follows from the explanations in Example 18.4. Therefore, G_1^{rs} also forms a dense open of G_1. Writing $j_1 : G_1^{\mathrm{rs}} \to G_1$ for the inclusion, we are led to consider

$$A_{\chi,1} = j_{1,!*}\mathcal{L}_{\chi,1}\langle \dim G\rangle.$$

These are simple, G_1-equivariant perverse sheaves on G_1, which are mixed but not necessarily pure. Lusztig discovered that they bear a close analogy with the unipotent principal series characters of G^F. To describe it, we tie this story back to the Hecke category.

Recall that a correspondence between varieties X and Y is a diagram of varieties of the form $X \leftarrow Z \to Y$. We define the *horocycle correspondence* to be the diagram of G_1-equivariant morphisms

$$\mathcal{B} \times \mathcal{B} \xleftarrow{act} G \times \mathcal{B} \xrightarrow{\pi} G,$$

where $act(g, B) = (gBg^{-1}, B)$ and $\pi(g, B) = g$. The map π extends the projection map $\tilde{G} \to G$, so our notation remains consistent. We are led to consider the *character functor*

$$\mathsf{CH}_1 := \pi_{1,!}act_1^*\langle \dim G - 2\dim \mathcal{B}\rangle : \mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1) \to \mathsf{D}_{G_1}(G_1).$$

It turns out that CH_1 provides something close to a categorification of the cocenter map for $H_W(\mathsf{x})$, but in fact, contains even more information.

Recall our notations $O_{w,1}, j_{w,1}, \Delta_{w,1}, E_{w,1}$. For all $w \in W$, let

$$\begin{aligned} K_{w,1} &= \mathsf{CH}_1(\Delta_{w,1}) = \mathsf{CH}_1(j_{w,1,!}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}}\langle \dim O_w\rangle), \\ \bar{K}_{w,1} &= \mathsf{CH}_1(E_{w,1}) = \mathsf{CH}_1(IC_{O_{w,1}}\langle \dim O_w\rangle). \end{aligned}$$

The smoothness of act_1 means that the pullback act_1^* is perverse t-exact. Due to the adjunction $act_{1,!} \vdash act_1^! = act_1^*\langle \dim B\rangle$, it also preserves (semi)simplicity. At the same time, by the decomposition theorem, the properness of π_1 means that the pushforward $\pi_{1,*}$ sends any mixed simple perverse sheaf E_1 to a mixed complex isomorphic, after pullback from G_1 to G, to a direct sum of shifts of simple perverse sheaves: more precisely, that $\pi_* E \simeq \bigoplus_i {}^p\mathcal{H}^i(E)[-i]$ with each term ${}^p\mathcal{H}^i(E)$ semsimple. Thus:

$$\bar{K}_w \simeq \bigoplus_i {}^p\mathcal{H}^i(\bar{K}_w)[-i]$$

with each term ${}^p\mathcal{H}^i(\bar{K}_w)$ semisimple. This suggests that even before pullback from G_1 to G, the sum $\bigoplus_i {}^p\mathcal{H}^i(\bar{K}_{w,1})[-i]$ might provide a *semisimplification* of $\bar{K}_{w,1}$.

Let $\mathsf{M}_G \subseteq \mathsf{D}_{G_1}(G_1)$ be the full additive subcategory generated by shift-twists of mixed objects of $\mathsf{Perv}_{G_1}(G_1)$. As with $[\mathsf{C}_W]_\oplus$, we regard $[\mathsf{M}_G]_\oplus$ as a $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-module on which x acts by $\langle -1 \rangle$. We will state a mysterious identity in $[\mathsf{M}_G]_\oplus$, discovered by Lusztig, that connects the objects $\bar{K}_{w,1}$ and $A_{\chi,1}$ to the very different geometric setting of Deligne–Lusztig theory.

18.5.

To this end, it is convenient to introduce an intersection-cohomology analogue of the unipotent Deligne–Lusztig virtual characters

$$R_w := \sum_i (-1)^i \mathrm{H}^i_c(X_w, \bar{\mathbf{Q}}_\ell).$$

Here, recall that $X_w = \{B \in \mathcal{B} \mid B \xrightarrow{w} FB\}$. Let $\bar{X}_w \subseteq \mathcal{B}$ be the Zariski closure of X_w, and let

$$\bar{R}_w(\mathsf{x}) = \sum_i (-\mathsf{x})^i \mathrm{H}^i_c(\bar{X}_w, IC_{\bar{X}_w}).$$

For all $\chi \in \mathrm{Irr}(W)$, let $\chi_\mathsf{x} : KH_W \to K$ be the trace that corresponds to $\chi : KW \to K$ under Tits deformation. Let

$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) R_w,$$

and let $\rho_\chi \in \mathrm{Irr}(G^F)$ be the unipotent irreducible character indexed by χ. Recall that this means

$$R_e = \bigoplus_\chi \rho_\chi \otimes \chi_\mathsf{x}|_{\mathsf{x} \to q^{1/2}} \quad \text{as a } (G^F, H_W(q^{1/2}))\text{-bimodule.}$$

Finally, let $(c_w)_{w \in W}$ be the Kazhdan–Lusztig basis of $H_W(\mathsf{x})$.

The following statement combines a result from Lusztig's book *Characters of Reductive Groups...*, as cited on page 67 of Carter's "On the Representations of the Finite Groups of Lie Type...", and a result extracted from Cor. 14.11 and Thm. 23.1 of Lusztig's "Character Sheaves" papers.

Theorem 18.5 (Lusztig)**.** *Let*

$$[\bar{K}_{w,1}] = \sum_i (-1)^i [{}^p\mathcal{H}^i(\bar{K}_{w,1})]$$

in $[\mathsf{M}_G]_\oplus$. *Then for all* χ, *we have*

$$\mathsf{x}^{-\ell(w)} (\bar{R}_w(\mathsf{x}), \rho_\chi)_{G^F, \mathsf{x}} = \sum_{\psi \in \mathrm{Irr}(W)} (R_\psi, \rho_\chi)_{G^F} \psi_\mathsf{x}(c_w) = ([\bar{K}_{w,1}] : [A_{\chi,1}])_\mathsf{x},$$

where on the left-hand side, $(-,-)_{G^F,\mathsf{x}}$ *is the* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-linear extension of* $(-,-)_{G^F}$, *and on the right-hand side,* $(- : -)_\mathsf{x}$ *refers to* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-graded multiplicity.*

As far as I understand, Lusztig proved the right-hand equality case by case, after reduction to the setting where G is almost-simple. It is hoped that the $(\infty, 2)$-categorical methods of Gaitsgory, Rozenblyum, Varshavsky, *et al.* will provide a more conceptual proof.

An important warning: The left-hand equality does *not* say that the only irreducible characters of G^F occurring in $\bar{R}_w(\mathsf{x})$ take the form ρ_χ. Similarly, it is *not* true that the only Jordan–Hölder factors of the mixed perverse sheaves ${}^p\mathcal{H}^i\mathcal{H}^i(\bar{K}_{w,1})$ are the objects $A_{\chi,1}$.

In general, simple perverse sheaves occurring as Jordan–Hölder factors of the objects $\bar{K}_w$, *resp.* $\bar{K}_{w,1}$, are called (mixed, equivariant) *unipotent character sheaves*, *resp.* *mixed unipotent character sheaves*. Just as we define cuspidal irreducible characters of G^F to be those not occurring in any principal series, we define *cuspidal unipotent character sheaves* to be those not isomorphic to A_χ for any χ.

The objects $K_{w,1}$ are more troublesome, since the objects $\Delta_{w,1}$ are not semisimple. Nonetheless, in an appropriate split Grothendieck group, it turns out that the change of basis from the classes $[{}^p\mathcal{H}^i(K_{v,1})]$ to the classes $[{}^p\mathcal{H}^i(\bar{K}_{w,1})]$ is given by the Kazhdan–Lusztig polynomials $P_{v,w}(\mathsf{q})$, just like the change of basis from the classes $[\Delta_{v,1}]$ to the classes $[E_{w,1}]$ in $[\mathsf{H}_W]_\Delta = H_W(\mathsf{x})$. In particular:

Corollary 18.6 (Lusztig)**.** *Let*

$$[K_{w,1}] = \sum_i (-1)^i [{}^p\mathcal{H}^i(K_{w,1})].$$

Then for all χ, we have

$$\sum_{\psi \in \mathrm{Irr}(W)} (R_\psi, \rho_\chi)_{G^F} \psi_{\mathsf{x}}(\sigma_w) = ([K_{w,1}] : [A_{\chi,1}])_{\mathsf{x}}.$$

In particular, taking $\mathsf{x} \to 1$, we have

$$(R_w, \rho_\chi)_{G^F} = \sum_\psi (R_\psi, \rho_\chi)\psi(w) = ([K_w] : [A_\chi]),$$

where on the right-hand side, $(- : -)$ is the ungraded *multiplicity of A_χ in K_w.*