17.

Having introduced the Hecke category and Rouquier complexes, we explain, following Khovanov, a categorification of Jones–Ocneanu's HOMFLYPT Markov trace.

17.1.

Let $k = \bar{\mathbf{F}}_q$ and $k_1 = \mathbf{F}_q$. Let $G, \mathcal{B}, O_w, j_w, B, T$ and their analogues G_1, *etc.* over k_1 be the same as last time.

We continue to assume that G_1 is a split form, so that Frobenius acts trivially on the Weyl group W and its system of simle reflections S. Recall the braid group attached to W: If

$$W = \langle S \mid s^2 = e \text{ and } \overbrace{sts\cdots}^{m_{s,t}} = \overbrace{tst\cdots}^{m_{s,t}} \text{ for all } s,t \in S\rangle,$$

then the associated *(Artin–Tits) braid group* is

$$Br_W = \langle (\sigma_s)_{s\in S} \mid \overbrace{\sigma_s\sigma_t\sigma_s\cdots}^{m_{s,t}} = \overbrace{\sigma_t\sigma_s\sigma_t\cdots}^{m_{s,t}} \text{ for all } s,t \in S\rangle.$$

If $w = s_{i_1}\cdots s_{i_\ell}$ is a word in S of minimal length, then $\sigma_w = \sigma_{s_{i_1}}\cdots\sigma_{s_{i_\ell}}$ is a well-defined element of Br_W depending only on w. Using this fact, one can show that Br_W admits the presentation

$$Br_W = \langle (\sigma_w)_{w\in W} \mid \sigma_w\sigma_{w'} = \sigma_{ww'} \text{ whenever } \ell(w) + \ell(w') = \ell(ww')\rangle,$$

which in turn gives rise to the identification

$$H_W(\mathsf{x}) = \frac{\mathbf{Z}[\mathsf{x}^{\pm 1}]Br_W}{\langle(\sigma_s - \mathsf{x})(\sigma_s + \mathsf{x}^{-1}) \mid s \in S\rangle}.$$

The composition $Br_W \to H_W(\mathsf{x})^\times \xrightarrow{\sim} [\mathsf{H}_W]_\Delta^\times$ sends $\sigma_w^{\pm 1} \mapsto [\mathcal{R}_w^\pm]$ for all w, and braid composition to convolution. We deduce that the classes $[\mathcal{R}_w^\pm]$ satisfy the braid relations.

Something stronger is true. The key is the following result, implicit in the work of Iwahori and also used later by Broué–Michel, Deligne, Lusztig, *etc.*

Lemma 17.1. *Whenever $\ell(w) + \ell(w') = \ell(ww')$, the forgetful map*

$$O_{w,1} \times_{\mathcal{B}_1} O_{w',1} \to O_{ww',1}$$

is a (G_1-equivariant) isomorphism of $\mathbf{F}_q$-schemes, where the fiber product uses the right-hand factor of $O_{w,1}$ and the left-hand factor of $O_{w',1}$.

This result implies that under the same hypothesis $\ell(w) + \ell(w') = \ell(ww')$, we have explicit isomorphisms in $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ of the form

$$\Delta_{w,1} * \Delta_{w',1} \xrightarrow{\sim} \Delta_{ww',1} \quad \text{and} \quad \nabla_{w,1} * \nabla_{w',1} \xrightarrow{\sim} \nabla_{ww',1}.$$

Now recall the weight realization functor $\mathsf{real} : \mathsf{H}_W(\mathsf{x}) \to \mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$. Since real is fully faithful, monoidal, and sends $\mathcal{R}_w^+ \mapsto \Delta_{w,1}$ and $\mathcal{R}_w^- \mapsto \nabla_{w,1}$, we deduce that whenever $\ell(w) + \ell(w') = \ell(ww')$, we have explicit isomorphisms

$$\mathcal{R}_w^+ * \mathcal{R}_{w'}^+ \xrightarrow{\sim} \mathcal{R}_{ww'}^+ \quad \text{and} \quad \mathcal{R}_w^- * \mathcal{R}_{w'}^- \xrightarrow{\sim} \mathcal{R}_{ww'}^-.$$

In this sense, Rouquier complexes satisfy categorified braid relations under convolution. To be more accurate, we have only shown this statement for the $\mathcal{R}_w^+$ and the $\mathcal{R}_w^-$ separately. We should also check that:

Lemma 17.2. *For all $s \in S$, there are explicit isomorphisms*

$$\mathcal{R}_s^+ * \mathcal{R}_s^- \xleftarrow{\sim} \underline{E_{e,1}} \xrightarrow{\sim} \mathcal{R}_s^- * \mathcal{R}_s^+,$$

where $\underline{E_{e,1}}$ is the complex consisting of $E_{e,1}$ in degree zero.

These isomorphisms do not come from isomorphisms of varieties, effectively because they mix together ! and $*$. Instead, they are checked using Soergel's embedding of C_W into the category of finitely-generated graded R-bimodules for $R = \mathrm{H}_B^*(pt) = \mathrm{H}_T^*(pt)$. All in all:

Theorem 17.3. *For any sequences $\vec{w} \in W^m$ and $\vec{\epsilon} \in \{\pm\}^m$, let*

$$\begin{aligned}
\beta_{\vec{w}} &= \sigma_{w_1} \cdots \sigma_{w_m}, \\
\beta_{\vec{w},\vec{\epsilon}} &= \sigma_{w_1}^{\epsilon_1} \cdots \sigma_{w_m}^{\epsilon_m}, \\
O_{\vec{w},1} &= O_{w_1,1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} O_{w_m,1}, \\
\mathcal{R}_{\vec{w},\vec{\epsilon}} &= \mathcal{R}_{w_1}^{\epsilon_1} * \cdots * \mathcal{R}_{w_m}^{\epsilon_m}.
\end{aligned}$$

Then:

(1) $O_{\vec{w}}, 1$ only depends on $\beta_{\vec{w}}$ up to isomorphism over $\mathcal{B} \times \mathcal{B}$, in the sense that if $\beta_{\vec{w}} = \beta_{\vec{w}'}$, then any explicit sequence of relations in the braid group producing the identity gives rise to an explicit isomorphism $O_{\vec{w},1} \simeq O_{\vec{w}',1}$ preserving the leftmost and rightmost projections to $\mathcal{B}$.

(2) $\mathcal{R}_{\vec{w},\vec{\epsilon}}$ only depends on $\beta_{\vec{w},\vec{\epsilon}}$ up to isomorphism, in the sense that if $\beta_{\vec{w},\vec{\epsilon}} = \beta_{\vec{w}',\vec{\epsilon}'}$, then any explicit sequence of relations in the braid group producing the identity gives rise to an explicit isomorphism $\mathcal{R}_{\vec{w},\vec{\epsilon}} \xrightarrow{\sim} \mathcal{R}_{\vec{w}',\vec{\epsilon}'}$.

When $\vec{\epsilon}$ consists solely of $+$'s, *resp.* solely of $-$'s, it will be convenient to write $\mathcal{R}_{\vec{w}}^+$, *resp.* $\mathcal{R}_{\vec{w}}^-$, in place of $\mathcal{R}_{\vec{w},\vec{\epsilon}}$. In this case, we also say that $\beta_{\vec{w},\vec{\epsilon}}$ is *positive*, *resp.* *negative*. (Thus $\beta_{\vec{w}}$ is positive.)

Remark 17.4. The original isomorphisms of varieties satisfy a strict form of associativity. Namely, the following square commutes on the nose:

$$\begin{array}{ccc} O_{w,1} \times_{\mathcal{B}_1} O_{w',1} \times_{\mathcal{B}_1} O_{w'',1} & \longrightarrow & O_{w,1} \times_{\mathcal{B}_1} O_{w'w'',1} \\ \downarrow & & \downarrow \\ O_{ww',1} \times_{\mathcal{B}_1} O_{w'',1} & \longrightarrow & O_{ww'w''} \end{array}$$

It implies similar associativity identities with $\Delta_{-,1}, \nabla_{-,1}, \mathcal{R}^+, \mathcal{R}^-$ in place of $O_{-,1}$. Using these identities, it is possible to describe more precisely the sense in which the isomorphisms of the form $O_{\vec{w},1} \xrightarrow{\sim} O_{\vec{w}',1}$ and $\mathcal{R}^{\pm}_{\vec{w}} \xrightarrow{\sim} \mathcal{R}^{\pm}_{\vec{w}'}$ are unique. Note that it suffices to handle the case where $\vec{w}, \vec{w}'$ are words in S. See Deligne, "Action du group des tresses sur un catégorie", for details.

17.2.

The discussion above leads us to regard Rouquier complexes as *categorified* braids. This in turn hints at categorifications of the braid and link invariants that we studied earlier, to be constructed from these complexes.

First observe that there is a naive analogue in algebraic geometry to forming the link closure of a (positive or negative) braid. For convenience below, we drop the subscript $_1$'s that indicate $\mathbf{F}_q$-structures. If

$$O_{\vec{w}} = \{(B_0 \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m) \in \mathcal{B}^{1+m}\}$$

represents such a braid itself, then

$$X_{\vec{w}} = \{(B_1, \ldots, B_m) \in \mathcal{B}^m \mid B_m \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m\}$$

the pullback of $O_{\vec{w}}$ along the diagonal $O_e = \mathcal{B} \to \mathcal{B} \times \mathcal{B}$, represents its cyclic closure. A more sophisticated construction: Form

$$G_{\vec{w}} = \{(g, B_1, \ldots, B_m) \in G \times \mathcal{B}^m \mid gB_m g^{-1} \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m\},$$

the pullback of $O_{\vec{w}}$ along the action map

$$\begin{aligned} &G \times \mathcal{B} \xrightarrow{act} \mathcal{B} \times \mathcal{B}, \\ &(g, B) \mapsto (gBg^{-1}, B). \end{aligned}$$

Everything here remains G-equivariant once we require that G acts on itself by left conjugation. Even though the definition of $G_{\vec{w}}$ does not have cyclic symmetry with respect to the coordinates $B_1, \ldots, B_m$, this symmetry is restored at the level of the quotient stack $[G \backslash G_{\vec{w}}]$.

17.3.

We now discuss how the passage from $O_{\vec{w}}$ to $G_{\vec{w}}$ looks at the level of Rouquier complexes. Recall from Soergel that the hypercohomology functor

$$\mathsf{H}^*_G(\mathcal{B}\times\mathcal{B},-) : \mathsf{H}_W = \mathsf{K}^b(\mathsf{C}_W) \to \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}})$$

is fully faithful, and that the R-bimodule structure arises from the identification

$$[G\backslash(\mathcal{B}\times\mathcal{B})] \simeq [B\backslash G/B].$$

Pulling back along *act* on the left-hand side corresponds to pulling back along

$$[G/\mathrm{Ad}(B)] \to [B\backslash G/B]$$

on the right-hand side, where $\mathrm{Ad}(B)$ connotes B acting on G by left conjugation, not by multiplication.

Therefore, the effect of pulling back along *act before* G-equivariant hypercohomology corresponds to replacing the $*$-pushforward to $[B\backslash pt/B]$ with the $*$-pushforward to $[pt/\mathrm{Ad}(B)]$. If we were working at the underived level, then this would correspond to base change from $(R\otimes R^{\mathrm{op}})$-modules to R-modules along the map $R\otimes R^{\mathrm{op}}\to R$ that sends $f_1\otimes f_2\mapsto f_1f_2$.

But since our sheaf operations happen at the derived level, we need to take the *derived* base change from $R\otimes R^{\mathrm{op}}$ to R. In more classical language, this is the *Hochschild homology* functor

$$\mathrm{HH}^* := \mathrm{Tor}_*^{R\otimes R^{\mathrm{op}}}(R,-).$$

To extend this operation from R-bimodules to complexes of R-bimodules up to homotopy, we just apply HH^* term by term to the complex. We will again write HH^* for the resulting functor $\mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}) \to \mathsf{K}^b(\mathsf{Vect}^{2\text{-}\mathsf{gr}}_{\bar{\mathbf{Q}}_\ell})$, where in general, $(-)^{d\text{-}\mathsf{gr}}$ will denote a $\mathbf{Z}^d$-grading.

To state the precise relation between Hochschild homology and G-quivariant hypercohomology over $\mathcal{B}\times\mathcal{B}$, we need to account somehow for the new, second *Hochschild grading*. Webster–Williamson showed that it has to do with weights. Recall that if $K_1\in\mathsf{D}_{G_1}(X_1)$ is a mixed complex, then $\mathrm{H}^*_G(X,K)$ forms a $\bar{\mathbf{Q}}_\ell[F]$-module. We define a *weight filtration* $\mathsf{W}_{\le *}$ on any such module V by setting $\mathsf{W}_{\le\alpha}V$ to be the span of the F-eigenvectors in V with eigenvalue λ such that $|\lambda|\le q^{\alpha/2}$ under any isomorphism $\bar{\mathbf{Q}}_\ell\simeq\mathbf{C}$.

Theorem 17.5 (Webster–Williamson)**.** *The functor* $\mathrm{H}^*_G(G\times\mathcal{B}, act^*(-))$*, where we first pull back along act before taking hypercohomology, factors as the composition*

$$\mathsf{H}_W \xrightarrow{\mathrm{H}^*_G(\mathcal{B}\times\mathcal{B},-)} \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}) \xrightarrow{\mathrm{HH}^*} \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R})$$

followed by the regrading

$$\mathrm{gr}^{R}_{i+j}\,\mathrm{HH}^{i}(\mathbf{B}_w\langle m\rangle) = \mathrm{gr}^{\mathsf{W}}_{i+j}\,\mathrm{H}^{j}_{G}(G \times \mathcal{B}, act^{*}E_w\langle m\rangle),$$

where on the left, gr^{R}_{*} *is the grading coming from the Soergel bimodule, and on the right,* $\mathrm{gr}^{\mathsf{W}}_{*}$ *is the* weight grading.

We define *triply-graded Khovanov–Rozansky homology* to be the functor on H_W given by the composition

$$\text{(17.1)}\quad \mathsf{HHH} : \mathsf{H}_W \xrightarrow{\mathrm{H}^{*}_{G}(\mathcal{B}\times\mathcal{B},-)} \mathsf{K}^{b}(\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}) \xrightarrow{\mathrm{HH}^{*}} \mathsf{K}^{b}(\mathsf{Vect}^{\text{2-gr}}_{\bar{\mathbf{Q}}_\ell}) \xrightarrow{\mathrm{H}_{*}} \mathsf{Vect}^{\text{3-gr}}_{\bar{\mathbf{Q}}_\ell}.$$

Building on the original work of Khovanov–Rozansky, Khovanov observed:

Theorem 17.6 (Khovanov)**.** *Suppose that* $W = S_n$*. Then after renormalizing and shifting the triple grading, the function on* $\mathsf{H}_{S_n}(\mathsf{x}) = [\mathsf{H}_{S_n}]_{\triangle}$ *induced by the Euler characteristic of* HHH *is the Jones–Ocneanu trace. Here we take the Euler characteristic with respect to the "Rouquier" grading, coming from the degree in the functor* H_{*} *in the last step of* (17.1)*.*

17.4.

We now calculate everything in the case of the identity object $E_{e,1}$. First, we have weight-preserving isomorphisms

$$\begin{aligned}\mathrm{H}^{j}_{G}(G \times \mathcal{B}, act^{*}E_e) &\simeq \mathrm{H}^{j}_{\mathrm{Ad}(B)}(B, \bar{\mathbf{Q}}_\ell)\\ &\simeq \mathrm{H}^{j}_{T}(T, \bar{\mathbf{Q}}_\ell)\\ &\simeq \bigoplus_{i} \mathrm{H}^{i}(T, \bar{\mathbf{Q}}_\ell) \otimes \mathrm{H}^{j-i}_{T}(pt, \bar{\mathbf{Q}}_\ell).\end{aligned}$$

Above, Frobenius acts by q^{i} on $\mathrm{H}^{i}(T)$ and by $q^{(j-i)/2}$ on $\mathrm{H}^{2d}_{T}(pt)$, so it acts by $q^{(i+j)/2}$ on the ith summand of the last expression. We deduce that

$$\mathrm{gr}^{\mathsf{W}}_{i+j}\,\mathrm{H}^{j}_{G}(G \times \mathcal{B}, act^{*}E_e) \simeq \mathrm{H}^{i}(T, \bar{\mathbf{Q}}_\ell) \otimes \mathrm{H}^{j-i}_{T}(pt, \bar{\mathbf{Q}}_\ell).$$

Compare this to

$$\mu_n(\sigma_e) = \mu_n(1) = \frac{\mathsf{a} - \mathsf{a}^{-1}}{\mathsf{x} - \mathsf{x}^{-1}} \cdot \mu_{n-1}(1) = \cdots = \left(\frac{\mathsf{a} - \mathsf{a}^{-1}}{\mathsf{x} - \mathsf{x}^{-1}}\right)^{n-1}.$$

(Recall that $\mu_1(1) = 1$.)