16.

We finally present a categorification of the Iwahori–Hecke algebra by means of mixed, equivariant perverse sheaves. In particular, we describe specific objects that categorify the elements of the standard and Kazhdan–Lusztig bases. Besides Achar, the main reference is Bezrukavnikov–Yun's paper "On Koszul Duality for Kac–Moody Groups".

16.1.

Like last time, $k = \bar{\mathbf{F}}_q$ and $k_1 = \mathbf{F}_q$ and ℓ is a prime invertible in k_1. Henceforth, we fix a square root $q^{1/2} \in \bar{\mathbf{Q}}_\ell^\times$. This allows us to define a *half Tate twist* $\bar{\mathbf{Q}}_\ell(\frac{1}{2})$. On any category of the form $\mathsf{D}_{G_1}(X_1)$, let

$$\langle m \rangle = (-) \otimes (\bar{\mathbf{Q}}_\ell)_{X_1}[m](\tfrac{m}{2}),$$

the so-called *shift-twist*. Since $[1]$, *resp.* $(\frac{1}{2})$, shifts weights up by 1, *resp.* down by 1, we know that $\langle 1 \rangle$ preserves the weights of mixed complexes.

Let G be a connected reductive algebraic group with $\mathbf{F}_q$-form G_1, and let $F : G \to G$ be the Frobenius corresponding to G_1. Let W be the Weyl group of G. For simplicity in what follows, we always assume that G_1 is the split form of G, so that F acts trivially on W. Let $\mathcal{B}_1$ be the flag variety of G_1.

For each $w \in W$, the orbit $j_w : O_w \to \mathcal{B} \times \mathcal{B}$ then arises from some locally closed $j_{w,1} : O_{w,1} \to \mathcal{B}_1 \times \mathcal{B}_1$. (Recall that $\dim O_w = \ell(w) + \dim \mathcal{B}$.) By a result of BBDG in the previous set of notes, the twists

$$E_{w,1} = j_{w,1,!*}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}} \langle \dim O_w \rangle$$

are all pure of weight 0. For all $m \in \mathbf{Z}$, let

$$\mathbf{B}_w\langle m \rangle = \mathrm{H}^*_G(\mathcal{B} \times \mathcal{B}, E_w\langle m \rangle),$$

the intersection cohomology of the pullback of $E_{w,1}\langle m \rangle$ from $\mathbf{F}_q$ to $k = \bar{\mathbf{F}}_q$, viewed as a $\bar{\mathbf{Q}}_\ell[F]$-module. We abbreviate by writing $\mathbf{B}_w$ for $\mathbf{B}_w\langle 0 \rangle$.

Here is where the use of equivariant objects, not merely orbit-constructible objects, becomes relevant. Fix an F-stable Borel $B \subseteq G$ corresponding to $B_1 \subseteq G_1$. Via the F-equivariant isomorphism of stacks

$$[G \backslash (\mathcal{B} \times \mathcal{B})] \simeq [B \backslash G / B],$$

we can identify $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1) \simeq \mathsf{D}_{B_1 \times B_1}(G_1)$. For any $K_1 \in \mathsf{D}_{B_1 \times B_1}(G_1)$, the hypercohomology $\mathrm{H}^*_{B \times B}(G, K)$ is endowed with an action of $\mathrm{H}^*_{B \times B}(pt, \bar{\mathbf{Q}}_\ell)$. Finally, by Künneth (for ordinary cohomology!),

$$\mathrm{H}^*_{B \times B}(pt, \bar{\mathbf{Q}}_\ell) \simeq R \otimes_{\bar{\mathbf{Q}}_\ell} R, \quad \text{where } R := \mathrm{H}^*_B(pt, \bar{\mathbf{Q}}_\ell).$$

Recall that if $T = B/[B, B]$, then $R \simeq \bar{\mathbf{Q}}_\ell[\mathrm{X}^*(T)]$ with $\mathrm{X}^*(T)$ placed in degree 2. In this way, $\mathbf{B}_w$ forms an R-bimodule for all w. Note that R and $\mathbf{B}_w$ are moreover cohomologically graded. From calculating the weights of F on $\mathrm{H}^*_T(pt, \bar{\mathbf{Q}}_\ell)$, one finds that $\langle 1\rangle$ shifts this grading by 1.

16.2.

The following results were essentially proved by Soergel in "Kategorie $\mathcal{O}$, Perverse Garben[,] und Moduln über den Koinvarianten zur Weylgruppe". See Section 3 of his paper, in particular. Note that Soergel works with $k = \mathbf{C}$ and the analytic, not étale, topology, and sidesteps the use of mixed structure with something more elementary. The mixed étale reformulations can be found in Bezrukavnikov–Yun Propositions 3.1.6 and 3.2.1.

Theorem 16.1 (Soergel). *(1) For any $w \in W$, the graded vector space $\mathbf{B}_w$ is free over R (up to grading shifts).*

(2) For any $v, w \in W$ and $m \in \mathbf{Z}$, we have

$$\mathrm{Hom}_{\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)}(E_{v,1}, E_{w,1}\langle m\rangle) \simeq \mathrm{Hom}_{\mathsf{Mod}^{\mathsf{gr}}_{R\otimes R^{\mathrm{op}}}}(\mathbf{B}_v, \mathbf{B}_w\langle m\rangle).$$

(3) The hypercohomology functor $\mathrm{H}^*_G(\mathcal{B} \times \mathcal{B}, -)$ transports the convolution $*$ on $\mathsf{D}_G(\mathcal{B} \times \mathcal{B})$ to the tensor product $\otimes_R$ on the category of finitely-generated graded R-bimodules $\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}$.

The following corollary, while inspired by Soergel's work, was first formalized in the mixed étale setting by Bezrukavnikov–Yun, in Proposition 3.2.5 and Remark 3.2.6 of their paper.

Corollary 16.2 (Bezrukavnikov–Yun). *(1) For $w \in W$ and $m \in \mathbf{Z}$, the F-action on $\mathbf{B}_w(m)$ is semisimple.*

(2) For any $v, w \in W$, the convolution $E_{v,1} * E_{w,1}$ is already *isomorphic to a (finite) direct sum of shifts of semisimple (equivariant) mixed perverse sheaves, before pullback from $\mathbf{F}_q$ to $\bar{\mathbf{F}}_q$. (This is stronger than the decomposition theorem!)*

In fact, this convolution is isomorphic to a sum of objects of the form $E_{x,1}\langle m\rangle$ for varying $x \in W$ and $m \in \mathbf{Z}$.

Let C_W be the full additive subcategory of $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ generated by the objects $E_{w,1}\langle m\rangle$ for all w, m. By Corollary 16.2, it is closed under convolution. By Theorem 16.1, we can even embed it fully faithfully into the category of graded R-bimodules, $\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}$, via an embedding that takes $*$ to $\otimes_R$.

We say that C_W is the *additive Hecke category*. Its essential image in $\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}$ is called the category of *Soergel bimodules*. We say that

$$\mathsf{H}_W := \mathsf{K}^b(\mathsf{C}_W)$$

is the *triangulated Hecke category*, or just the *Hecke category*. It inherits a convolution from C_W. Recall that $[\mathsf{H}_W]_\triangle \simeq [\mathsf{C}_W]_\oplus$. What follows is essentially Soergel's Zerlegungssatz 1.

Theorem 16.3 (Soergel)**.** *We have an isomorphism of* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-algebras*

$$[\mathsf{C}_W]_\oplus \simeq H_W(\mathsf{x}),$$

where x *acts on the left by* $\langle -1\rangle$*, the multiplication on the left is induced by* $*$*, and the multiplication on the right is induced by* $*$*. Under this isomorphism,* $[E_w]$ *corresponds to the Kazhdan–Lusztig element* c_w*, for all* $w \in W$*.*

16.3.

How about the standard basis elements σ_w? In the category C_W, there are actually many objects whose class in $[\mathsf{C}_W]_\oplus$ corresponds to σ_w. But it turns out that from a geometric viewpoint, there is a best choice. Recall that the intermediate extension $j_{w,!*}$ is so named because it is intermediate between $j_{w,!}$ and $j_{w,*}$. It turns out that $j_{w,!}$, the extension by zero, has the flavor of σ_w, whereas $j_{w,*}$ has the flavor of σ_w^{-1}. To this end, let

$$\Delta_{w,1} = j_{w,1,!}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}}\langle \dim O_w\rangle \quad \text{and} \quad \nabla_{w,1} = j_{w,1,*}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}}\langle \dim O_w\rangle.$$

These objects are also perverse sheaves, due to the following exactness properties proved in BBDG: See their Proposition 2.2.5 and §4.1.

Theorem 16.4 (BBDG)**.** *For the perverse* t*-structure and any map* $p : Y \to X$ *between separated schemes of finite type over a field:*

(1) *If* p *is quasi-finite, then* $p_!$ *is right* t*-exact and* p_* *is left* t*-exact.*
(2) *If* p *is affine, then* $p_!$ *is left* t*-exact and* p_* *is right* t*-exact.*

In particular, if p *is composed of any finite map followed by an affine open embedding, then* $p_!$ *and* p_* *are both perverse* t*-exact.*

We immediately face a problem: $\Delta_{w,1}$ and $\nabla_{w,1}$ do not actually live in C_W. That is, they cannot be decomposed into sums of shifts of the objects $E_{w,1}$. The solution comes from BBDG Théorème 5.3.5:

Theorem 16.5 (BBDG)**.** *Every mixed perverse sheaf* E_1 *on a scheme of finite type over* k_1 *admits a unique, finite, increasing* weight filtration $\mathsf{W}_{\leq *}$*, such that* $\mathrm{gr}_i^{\mathsf{W}} E_1$ *is pure of weight* i *for all* i*.*

This theorem extends to the G_1-equivariant setting. In fact, we can describe the weight filtrations explicitly in the case of simple reflections. Everything in the discussion that follows will be F-stable, so for convenience, we will drop the subscript $_1$ everywhere.

Let $S \subseteq W$ be the set of simple reflections. For any $s \in S$, we have $\bar{O}_s = O_e \cup O_s$. For convenience, we write $i : O_e \to \bar{O}_s$ and $j : O_s \to \bar{O}_s$ for the inclusion maps, which are closed and open, respectively. The structure of these varieties and maps can be reduced to the rank-1 case. To explain how, we recall the fixed F-stable Borel $B = T \ltimes [B, B]$. We identify $W = N_G(T)/T$ and set $P_s = B \cup BsB$, a parabolic subgroup of G. Let L_s be the Levi of P_s, so that $P_s = L_s \ltimes [P_s, P_s]$. Then we can identify P_s/B with the flag variety of L_s, a copy of $\mathbf{P}^1$, and $BsB/B \subseteq P_s/B$ with a copy of $\mathbf{A}^1$. The left B-action on these varieties factors through a $(L_s \cap B)$-action. At the same time, we have isomorphisms of stacks

$$\begin{aligned}
[G\backslash \bar{O}_s] &\simeq [B\backslash P_s/B] \simeq [B\backslash \mathbf{P}^1],\\
[G\backslash O_e] &\simeq [B\backslash pt],\\
[G\backslash O_s] &\simeq [B\backslash BsB/B] \simeq [B\backslash \mathbf{A}^1].
\end{aligned}$$

We deduce that at the equivariant level, the maps i and j are comparable to the inclusion maps $pt \to \mathbf{P}^1$ and $\mathbf{A}^1 \to \mathbf{P}^1$. In particular,

$$E_s = j_{s,!*}(\bar{\mathbf{Q}}_\ell)_{O_s}\langle \dim O_s\rangle \simeq (\bar{\mathbf{Q}}_\ell)_{\bar{O}_s}\langle \dim \bar{O}_s\rangle$$

since $\mathbf{P}^1$ is smooth.

Recall that in the Hecke algebra, $c_s = \sigma_s + \mathsf{x}^{-1} = \sigma_s^{-1} + \mathsf{x}$. The corresponding identities in $[\mathsf{C}_W]_\oplus$ are

$$[E_{s,1}] = [\Delta_{s,1}] + [E_{e,1}\langle 1\rangle] = [\nabla_{s,1}] + [E_{e,1}\langle -1\rangle].$$

Granting that these identities do not arise from mere direct sum decompositions, we are led to look for extensions between the mixed complexes. From the exact triangle $j_! j^* \to \mathrm{id} \to i_! i^* \to$ and our description of $E_{s,1}$ as a shift-twisted constant sheaf, we obtain an exact triangle

$$\Delta_{s,1} \to E_{s,1} \to E_{e,1}\langle 1\rangle \to$$

whose Verdier dual is

$$\to E_{e,1}\langle -1\rangle \to E_{s,1} \to \nabla_{s,1}.$$

These triangles describe the weight filtrations on $\Delta_{s,1}$ and $\nabla_{s,1}$ (after rotation), since $E_{s,1}, E_{e,1}$ are pure of weight zero. Note that $\Delta_{s,1} \to E_{s,1}$ and $E_{s,1} \to \nabla_{s,1}$ are precisely the morphisms that arise from the definition of $j_{s,1,!*}$.

16.4.

In the triangulated Hecke category $\mathsf{H}_W = \mathsf{K}^b(\mathsf{C}_W)$, we set

$$\mathcal{R}_s^+ = (\underline{E_{s,1}} \to E_{e,1}\langle 1\rangle) \quad \text{and} \quad \mathcal{R}_s^- = (E_{e,1}\langle -1\rangle \to \underline{E_{s,1}}),$$

where as before, the underlining indicates the terms in degree zero. More generally, if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression—a word in S for w of minimal length—then we set

$$\mathcal{R}_w^+ = \mathcal{R}_{s_{i_1}}^+ * \cdots * \mathcal{R}_{s_{i_\ell}}^+ \quad \text{and} \quad \mathcal{R}_w^- = \mathcal{R}_{s_{i_1}}^- * \cdots * \mathcal{R}_{s_{i_\ell}}^-$$

These objects, or rather their images in $\mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R \otimes R^{\mathrm{op}}})$ under the functor that takes equivariant hypercohomology term by term, are now called *Rouquier complexes*. Rouquier discussed their Soergel-bimodule incarnations in, *e.g.*, his paper "Categorification of $\mathfrak{sl}_2$ and Braid Groups".

The following result is essentially shown in Appendix B of Bezrukavnikov–Yun's paper, where they refer to (some version of) H_W as a "dg-model".

Theorem 16.6. *There is a* weight realization functor

$$\mathsf{real} : \mathsf{H}_W \to \mathsf{D}^b_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$$

such that:

(1) real *is full, faithful, additive, and monoidal (*i.e.*, convolution-preserving).*
(2) real *restricts to the identity functor on* C_W*, viewed as the full subcategory of* H_W *of complexes supported in degree zero.*
(3) $\mathsf{real}(\mathcal{R}_w^+) = \Delta_{w,1}$ *and* $\mathsf{real}(\mathcal{R}_w^-) = \nabla_{w,1}$ *for all* w.

Corollary 16.7. *Under* $[\mathsf{H}_W]_\triangle \simeq H_W(\mathsf{x})$*, the class* $[\mathcal{R}_w^\pm]$ *corresponds to* $\sigma_w^{\pm 1}$.

Proof. We have checked the statement for simple reflections $s \in S$. The corresponding elements σ_s generate $H_W(\mathsf{x})$, so using the properties of real above, we can extend the statement from S to W. □

We conclude with an observation about the Verdier duality functor $\mathbf{D} = \mathbf{D}_{\mathcal{B}_1 \times \mathcal{B}_1}$. By construction, $\mathsf{C}_W \subseteq \mathsf{D}^b_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ is stable under $\mathbf{D}$. By contrast, $\mathbf{D}$ interchanges $\Delta_{s,1}$ and $\nabla_{s,1}$ for all $s \in S$. Similarly, the involutive functor on H_W induced by $\mathbf{D}$ interchanges $\mathcal{R}_s^+$ and $\mathcal{R}_s^-$. In this way, we see that the involution induced by $\mathbf{D}$ on $[\mathsf{C}_W]_\oplus$ corresponds to the involution D on $H_W(\mathsf{x})$, which sends $\mathsf{x} \mapsto \mathsf{x}^{-1}$ and $\sigma_s \mapsto \sigma_s^{-1}$.