5.

Throughout, G is a connected, smooth reductive algebraic group over $k = \bar{\mathbf{F}}_q$ with a Frobenius map $F : G \to G$. We fix an F-stable Borel pair (B, T) and write $U = [B, B]$. We fix $\delta \geq 1$ so that F^δ acts trivially on $W = N_G(T)/T$, and a section $w \mapsto \dot{w} : W \to N_{G^{F^\delta}}(T^{F^\delta})$. With these choices, $X_w \subseteq G/B$ and $\tilde{X}_w \subseteq G/U$ are F^δ-stable for all $w \in W$.

5.1.

Recall that in our running example where $G = \mathrm{SL}_2$ and F is standard, we can write $W = \{e, s\}$ with $e = \mathrm{id}$, and take $\delta = 1$. Last time, we computed the graded $\bar{\mathbf{Q}}_\ell[F]$-modules formed by the compactly-supported ℓ-adic cohomologies of X_e and X_s:

$$\mathrm{H}_c^*(X_e) \simeq \bar{\mathbf{Q}}_\ell^{\oplus(q+1)}, \qquad \mathrm{H}_c^*(X_s) \simeq \bar{\mathbf{Q}}_\ell^{\oplus q}[-1] \oplus \bar{\mathbf{Q}}_\ell[-2](-1).$$

Above $[-m]$ means "shift up by degree m" and $(-m)$ means "twist the Frobenius action by a factor of q^m".

One more property of ℓ-adic cohomology that I could have added to the list from last time:

(10) $\mathrm{H}^0(X)$ is the vector space of $\bar{\mathbf{Q}}_\ell$-valued functions on the set of connected components of X.

This gives another way to identify $\mathrm{H}_c^*(X_e) = \mathrm{H}_c^0(X_e)$, and by Poincaré duality, $\mathrm{H}_c^2(X_s) \simeq \mathrm{H}^0(X_s)^\vee[-2](-1)$. But it does more: It enables us to identify the G^F-actions on these vector spaces. It remains for us to identify the G^F-action on $\mathrm{H}_c^1(X_s)$.

5.2.

As mentioned last time, it is easier in general to work with the virtual character $R_{w,\theta}$ than with the individual representations $\mathrm{H}_c^i(\tilde{X}_w)[\theta]$. For any k-scheme of finite type X and automorphism $g : X \to X$, the *Lefschetz number* of g on $\mathrm{H}_c^*(X)$ is defined to be

$$\mathcal{L}_X(g) = \sum_i (-1)^i \operatorname{tr}(g \mid \mathrm{H}_c^i(X)).$$

The Lefschetz fixed-point formula tells us that if f is a Frobenius map, then $\mathcal{L}_X(f) = |X^f|$. At the same time,

$$\mathcal{L}_{X_w} = R_{w,1},$$
$$\mathcal{L}_{\tilde{X}_w} = \sum_\theta R_{w,\theta}$$

as functions on G^F.

The next result that we present, combining Exercise 4.7.4 and Theorem 4.4.12 in Geck, is a bridge between these two uses of Lefschetz number. Recall that $g : X \to X$ commutes with a Frobenius map $F : X \to X$ corresponding to some $\mathbf{F}_q$-rational structure $X = X_1 \otimes k$ if and only if g descends to X_1, meaning $g = g_1 \otimes \mathrm{id}$. Note that since X is of finite type, g is cut out by finitely many polynomials in finitely many variables. Thus, g is always defined over some finite subfield of k; in other words, given g, we can always find some Frobenius that commutes with g.

Theorem 5.1. *Suppose that X is a smooth k-variety with Frobenius f, and $g : X \to X$ is an automorphism of finite order that commutes with f. Then:*

(1) gf^m is a Frobenius map on X for all $m \geq 1$.
(2) The formal series

$$\mathcal{L}_X(g,t) := -\sum_{m\geq 1} |X^{gf^m}| t^m$$

satisfies $\mathcal{L}_X(g) = \lim_{t\to\infty} \mathcal{L}_X(g,t)$.

Proof of (2) from (1). Since f and g commute, we can triangularize them simultaneously. Suppose that $(\lambda_{i,j})_j$, *resp.* $(\mu_{i,j})_j$, is the list of eigenvalues of f, *resp.* g, on $\mathrm{H}^i_c(X)$. Since gf^m is a Frobenius map, the Lefschetz formula gives

$$|X^{gf^m}| = \sum_i (-1)^i \sum_j \mu_{i,j}\lambda_{i,j}^m,$$

from which

$$-\mathcal{L}_X(g,t) = \sum_{m,i,j} (-1)^i \mu_{i,j}\lambda_{i,j}^m t^m = \sum_{i,j} (-1)^i \mu_{i,j} \frac{\mu_{i,j}t}{1-\mu_{i,j}t}.$$

Now observe that $\frac{\mu_{i,j}t}{1-\mu_{i,j}t} \to -1$ as $t \to \infty$. □

Remark 5.2. The *Weil zeta series* of X with respect to f is defined by

$$Z_X(t) = \exp\left(\sum_{m\geq 1} |X^{f^m}| \frac{t^m}{m}\right),$$

where exp is a formal exponential. We see that

$$\mathcal{L}_X(\mathrm{id},t) = -t\frac{d}{dt}\log Z_X(t).$$

In this sense, $\mathcal{L}(t, \mid g, X)$ is a mild generalization of the zeta series.

Corollary 5.3. *Keeping the hypotheses of Theorem 5.1, suppose that X is the union of disjoint subvarieties X' and X'' that are f-stable and g-stable. Then*

$$\mathcal{L}_X = \mathcal{L}_{X'} + \mathcal{L}_{X''}$$

as functions of g.

Previously, we sketched the reason why X_w and $\tilde{X}_w$ form smooth varieties. If $g \in G^F$, then the action of g on G/B and G/U commutes with that of F, and hence, its action on X_w and $\tilde{X}_w$ commutes with that of F^δ. So we can apply Theorem 5.1 and its corollary to the case where $X = X_w, \tilde{X}_w$, or some unions of these, and $f = F^\delta$ and $g \in G^F$.

Returning to the setup with $G = \mathrm{SL}_2$ and F standard, we deduce that

$$\mathcal{L}_{G/B} = \mathcal{L}_{X_e} + \mathcal{L}_{X_s} = R_{e,1} + R_{s,1}.$$

We also know the cohomology of G/B, since it is $\mathbf{P}^1$:

$$\mathrm{H}^*_c(G/B) \simeq \mathrm{H}^*(G/B) \simeq \bar{\mathbf{Q}}_\ell \oplus \bar{\mathbf{Q}}_\ell[-2](-1),$$

Since $\mathrm{H}^0(G/B)$ carries the trivial representation of G^F, the same is true of its Poincaré dual $\mathrm{H}^2(G/B)$. Therefore $\mathcal{L}_{G/B}(g) = 2$ for all g.

From Mackey, we saw that the G^F-equivariant endomorphisms of $\mathbf{R}_{e,1} = \mathrm{H}^*_c(X_e) = \mathrm{H}^0_c(X_e)$ form a 2-dimensional algebra, which forces $\mathbf{R}_{e,1}$ to be a sum of two irreducible representations of G^F. But $\mathbf{R}_{e,1}$ is also the space of functions on X_e, which contains the trivial representation. So we must have

$$R_{e,1} = 1 + \mathrm{St} \quad \text{for some irreducible character St.}$$

This is the Steinberg character mentioned previously. Finally,

$$R_{s,1} = \mathcal{L}_{G/B} - R_{e,1} = 2 - (1 + \mathrm{St}) = 1 - \mathrm{St}.$$

Since $\mathbf{R}_{s,1} = \mathrm{H}^1_c(X_s) \oplus \mathrm{H}^2_c(X_s)$, and $\mathrm{H}^2_c(X_s)$ also carries the trivial character, we deduce that $\mathrm{H}^1_c(X_s)$ carries the Steinberg character.

5.3.

Before we can describe $\mathbf{R}_{s,\theta} = \mathrm{H}^*_c(X_s)[\theta]$ and $R_{s,\theta}$ for other θ, we should describe T^{sF} more explicitly. Taking T to be the diagonal torus given by

$$T(k) = \{t_a \mid a \in k^\times\}, \quad \text{where } t_a = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix},$$

we see that $s \cdot t_a = t_{a^{-1}}$. Therefore,

$$T^{sF} = \{t_a \in T \mid a^q = a^{-1}\} = \{t_a \in T \mid a^{q+1} = 1\}.$$

In particular, T^{sF} is cyclic of order $q + 1$.

Note that the condition $a^{q+1} = 1$ forces $a \in \mathbf{F}_{q^2}^\times$. Moreover, $a \in \mathbf{F}_q^\times$ only happens for $a = \pm 1$. These computations show that in general, the embedding of T into G does *not* restrict to an embedding of T^{sF} into G^F.

5.4.

Nonetheless, it turns out that there is another F-stable maximal torus $S \subseteq G$ such that T^{sF} is conjugate to $S^F \subseteq G^F$. It is convenient to explain a generalization of this fact to arbitrary G and T. In fact, we only need T to be F-stable, not necessarily contained in an F-stable Borel, for what follows.

Let $L : G \to G$ be the Lang map $L(g) = g^{-1}F(g)$, and let $\mathcal{T}_{G,F}$ be the set of F-stable maximal tori in G. There are maps

$$W \xleftarrow{L(-)/T} L^{-1}(N_G(T))(k) \xrightarrow{g \mapsto gTg^{-1}} \mathcal{T}_{G,F}.$$

Note that if $S \in \mathcal{T}_{G,F}$ satisfies $S = gTg^{-1}$ for some $g \in G(k)$, then by the F-stability of S, we require $g \in L^{-1}(N_G(T))(k)$. Thus the rightward map is surjective, just like the leftward map. Moreover:

(1) If $w \in W$ and $S \in \mathcal{T}_{G,F}$ admit a common lift $g \in L^{-1}(N_G(T))(k)$, then the identity $S = gTg^{-1}$ restricts to

$$S^F = gT^{wF}g^{-1}.$$

In particular, starting from w, *resp.* S, we can produce some S, *resp.* w, and a common lift g that together satisfy the identity above.

(2) If $gTg^{-1} = g'T(g')^{-1}$ for some $g, g' \in L^{-1}(N_G(T))(k)$, and $w, w' \in W$ are the respective images of $L(g), L(g')$, then

$$w' = x^{-1}wF(x) \quad \text{for some } x \in W. \tag{5.1}$$

Namely, take x to be the image of $g^{-1}g' \in N_G(T)(k)$.

In general, we say that elements $w, w' \in W$ are *F-conjugate* if and only if (5.1) holds. The discussion above shows that there is a well-defined map from $\mathcal{T}_{G,F}$ onto the set of F-conjugacy classes of W. The image of a torus under this map is sometimes called its *type*.

Proposition 5.4. *The map that sends an F-stable maximal torus to its type descends to a bijection*

$$\mathcal{T}_{G,F}/(G^F\text{-conjugacy}) \xrightarrow{\sim} W/(F\text{-conjugacy}).$$

Proof. The original map is surjective because the map $L^{-1}(N_G(T))(k) \to W$ is surjective, and factors through the G^F-conjugacy relation on $\mathcal{T}_{G,F}$ because if $g \in G(k)$ and $h \in G^F$, then $L(hg) = L(g)$.

It remains to show injectivity. Suppose that $g, g' \in L(N_G(T))(k)$, that w, w' are the respective images of $L(g), L(g')$, and that $w' = xwF(x)$ for some

$x \in W$. Lifting x to $\dot{x} \in N_G(T)$, we must have $L(g') = t^{-1}L(g\dot{x})$ for some $t \in T(k)$. Setting $h = g'g^{-1}$, we see that

$$L(h) = L(g'g^{-1}) = gt^{-1}\dot{x}^{-1}g^{-1}F(g\dot{x}g^{-1}) = (g\dot{x}g)^{-1}(gt'g^{-1})F(g\dot{x}g^{-1})$$

for some $t' \in T(k)$. Setting $h' = hg\dot{x}^{-1}g^{-1}$, we get $L(h') = gt'g^{-1}$. By Lang, we can find $z \in gT(k)g^{-1}$ such that $L(z) = gt'g^{-1}$. Setting $h'' = h'z^{-1}$, we see that $F(h'') = h''$ and $g'T(g')^{-1} = h''(gTg^{-1})(h'')^{-1}$, as needed. □

5.5.

To conclude our discussion of fixed-point formulas, we present two major results by Deligne–Lusztig, and explain their application to the discrete series of $\mathrm{SL}_2(\mathbf{F}_q)$. Geck omits their proofs in his Section 4.5.

The first result is Deligne–Lusztig Theorem 3.2. To motivate it, recall that any invertible matrix g over a field has a *Jordan decomposition* $g = g_s g_u = g_u g_s$, where g_s is diagonalizable (or *semisimple*) and g_u is unipotent. If the field characteristic is $p > 0$ and the (multiplicative) order of g is finite, then the orderof g_s is coprime to p, while the order of g_u is a power of p.

Theorem 5.5 (Deligne–Lusztig)**.** *Suppose that X is a smooth affine k-variety with Frobenius f, and $g : X \to X$ is an automorphism of finite order that commutes with f. Suppose that $g = g_s g_u = g_u g_s$, where $g_s : X \to X$,* resp. *$g_u : X \to X$, has order coprime to p,* resp. *a power of p. Then*

$$\mathcal{L}_X(g) = \mathcal{L}_{X^{g_s}}(g_u).$$

In the SL_2 example, this theorem implies that for any $t \in T^{sF}$, we have $\mathcal{L}_{\tilde{X}_s}(t) = \mathcal{L}_{\tilde{X}_s^t}(1)$. But T^{sF} acts freely on $\tilde{X}_s$, so the right-hand side vanishes whenever $t \neq 1$! By character theory, we deduce that as a representation of T^{sF}, the vector space $\mathrm{H}_c^*(\tilde{X}_s)$ is a $\oplus$-power of the regular representation of T^{sF}. Since T^{sF} is abelian, every character occurs in the latter with the same multiplicity. Therefore

$$\dim R_{s,\theta} = \dim R_{s,1} = 1 - q \quad \text{for all } \theta.$$

To actually determine how these characters decompose beyond the $\theta = 1$ case, we need more firepower.

The following result, Deligne–Lusztig Theorem 6.8, generalizes the orthogonality formula we obtained earlier from Mackey decomposition. To make sense of the statement, observe that if $T^{wF} = T^{w'F}$ for some $w, w' \in W$, so that w, w' are F-conjugate, and $x \in W$ satisfies $w' = x^{-1}wF(x)$, then $xT^{w'F}x^{-1} = T^{wF}$. Hence, there is a bijection from characters of $T^{w'F}$ to characters of T^{wF} given by $\theta' \mapsto {}^x\theta'(-) = \theta'(x^{-1}(-)x)$.

Theorem 5.6 (Deligne–Lusztig)**.** *For any $w, w' \in W$, and for any character θ of T^{wF},* resp. *θ' of $T^{w'F}$, we have*

$$(R_{w,\theta}, R_{w',\theta'})_{G^F} = |\{x \in W \mid w' = x^{-1}wF(x) \text{ and } \theta = {}^x\theta'\}|.$$

Corollary 5.7. *In the setup above,*

$$(R_{w,\theta}, R_{w,\theta})_{G^F} = |\{x \in W \mid w = x^{-1}wF(x) \text{ and } \theta = {}^x\theta\}|.$$

In the SL_2 example, we have

$$(R_{s,\theta}, R_{s,\theta})_{G^F} = \begin{cases} 2 & \theta^2 = 1, \\ 1 & \text{else.} \end{cases}$$

In particular, $-R_{s,\theta}$ is an actual, irreducible representation of G^F whenever θ is a character of T^{sF} such that $\theta^2 \neq 1$. For q odd, there are $q-1$ choices of such θ, which form $\frac{1}{2}(q-1)$ conjugate pairs under s. Each pair contributes one new irreducible. The remaining two irreducibles of G^F are the summands of $\mathbf{R}_{s,\theta}$ for θ the order-2 character of T^{sF}. Taken together, these are all the *discrete series representations* of $\mathrm{SL}_2(\mathbf{F}_q)$.