4.

Today we review étale cohomology as a black-box formalism, which will also serve as a warm-up for later lectures about the constructible derived category, then use étale cohomology to construct virtual characters of G^F from Deligne–Lusztig varieties. Besides the original paper of Deligne–Lusztig, I will follow Bonnafé's book and notes I took from a WARTHOG course by Dudas.

4.1.

Throughout, $[d]$ means the degree-d shift functor on $\mathbf{Z}$-graded vector spaces V, so that $(V[d])^i = V^{i+d}$ for all i.

Fix a prime ℓ invertible in k. For our purposes, the *ℓ-adic étale cohomology* of a scheme X of finite type over k consists of $\mathbf{Z}$-graded $\bar{\mathbf{Q}}_\ell$-vector spaces

$$\mathrm{H}^*(X) = \bigoplus_i \mathrm{H}^i(X) \quad \text{and} \quad \mathrm{H}^*_c(X) = \bigoplus_i \mathrm{H}^i_c(X)$$

satisfying these properties, where all maps of graded vector spaces are assumed to be grading-preserving:

(1) Any map $f : Y \to X$ induces

$$\text{a } \textit{pullback } f^* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y).$$

If f is smooth of relative dimension d, then it induces

$$\text{a } \textit{!-pushforward } f_! : \mathrm{H}^*_c(Y)[2d] \to \mathrm{H}^i_c(X).$$

Similarly, if f is proper, then it induces

$$\text{a } \textit{pushforward } f_! = f_* : \mathrm{H}^*_c(Y) \to \mathrm{H}^*_c(X).$$

All of these constructions are functorial in f. In particular, if a group Γ acts on X, then it acts on $\mathrm{H}^*(X)$ contravariantly. If Γ acts by proper maps, then it also acts on $\mathrm{H}^*_c(X)$ covariantly.

(2) There are functorial maps $\mathrm{H}^*_c(X) \to \mathrm{H}^*(X)$. They are isomorphisms for proper X.

(3) For X connected and smooth of dimension n, there is a perfect pairing

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*_c(X) \to \bar{\mathbf{Q}}_\ell[-2n].$$

called *Poincaré duality*. Note that the grading-preserving condition means that it restricts to a perfect pairing between $\mathrm{H}^i(X)$ and $\mathrm{H}^{2n-i}_c(X)$.

(4) For any closed embedding $i : Z \to X$ with complement $j : U \to X$, we have a long exact sequence

$$\cdots \to \mathrm{H}_c^*(U) \xrightarrow{j_!} \mathrm{H}_c^*(X) \to \mathrm{H}_c^*(Z) \to \mathrm{H}_c^*(U)[1] \to \cdots$$

When X is proper, so that Z is also proper, the map $\mathrm{H}_c^*(X) \to \mathrm{H}_c^*(Z)$ is dual via item (2) and Poincaré to the map $i_! = i_*$.

(5) Pullback induces functorial isomorphisms

$$\mathrm{H}^*(X \sqcup Y) \simeq \mathrm{H}^*(X) \oplus \mathrm{H}^*(Y) \quad \text{and} \quad \mathrm{H}^*(X \times Y) \simeq \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y),$$

and similarly with H_c^* in place of H^* (by Poincaré).

(6) For the affine n-space $\mathbf{A}^n$, we have

$$\begin{aligned} \mathrm{H}^*(\mathbf{A}^n) &\simeq \bar{\mathbf{Q}}_\ell \text{ (in degree zero)}, \\ \mathrm{H}_c^*(\mathbf{A}^n) &\simeq \bar{\mathbf{Q}}_\ell[-2n] \text{ (by Poincaré)}. \end{aligned}$$

(7) If $d = \dim X$, then $\mathrm{H}^i(X) = 0$ for $i > 2d$ and $i < 0$. If X is moreover affine, then $\mathrm{H}_c^i(X) = 0$ for $i < d$.

We say that $\mathrm{H}^*(X)$ is the *ordinary cohomology* and $\mathrm{H}_c^*(X)$ the *compactly-supported cohomology*.

Now instead of schemes of finite type over k, consider the category of pairs (X, F), where X is of finite type over k and $F : X \to X$ is a Frobenius map corresponding to an $\mathbf{F}_q$-rational structure on X, where morphisms of such pairs are the k-morphisms that commute with the Frobenius maps.

Let $\bar{\mathbf{Q}}_\ell(m)$ be the *m-fold Tate twist*: the one-dimensional representation of $\langle F \rangle$ given by $F \cdot 1 = q^{-m}$. Then:

(8) The maps in items (1)–(6) are F-equivariant after we replace $[2m]$ with $[2m](m)$.

(9) For smooth X, we have the *Lefschetz fixed-point formula*

$$|X^F| = \sum_i \mathrm{tr}(F \mid \mathrm{H}_c^i(X)).$$

Note that the right-hand side uses H_c^i, not H^i.

Example 4.1. The formula for the ℓ-adic cohomology of affine space implies the formula for that of projective space, via Lefschetz. First, use the partition $\mathbf{P}^n = \mathbf{A}^n \sqcup \mathbf{P}^{n-1}$ and induction to show that $\mathrm{H}^i(\mathbf{P}^n)$ vanishes for i odd and that F acts on $\mathrm{H}^{2j}(\mathbf{P}^n)$ by q^j. Next, since $|\mathbf{P}^n(\mathbf{F}_q)| = 1 + q + \cdots + q^n$, Lefschetz forces $\dim \mathrm{H}^{2j} = 1$ for $0 \leq j \leq n$.

4.2.

Let G be a connected, reductive algebraic group over $k = \bar{\mathbf{F}}_q$ with Weyl group W, and let $F : G \to G$ be a Frobenius map. Last time we defined the varieties X_w and $\tilde{X}_w$. Let us now present a slightly different viewpoint on X_w.

Recall that $\mathcal{B}$ is the flag variety of G, isomorphic to G/B for any choice of Borel B, but itself independent of that choice. Let $O_w \subseteq \mathcal{B} \times \mathcal{B}$ be the G-orbit indexed by $w \in W$. Explicitly, if we fix a Borel B, then the k-points of O_w are the pairs (gB, gwB) for $g \in G(k)$. We see that X_w can be defined through a cartesian square:

$$\begin{array}{ccc} X_w & \longrightarrow & \mathcal{B} \\ \big\downarrow & & \big\downarrow {\scriptstyle F \times \mathrm{id}} \\ O_w & \longrightarrow & \mathcal{B} \times \mathcal{B} \end{array}$$

It turns out that O_w is smooth of dimension $\ell(w) + \dim \mathcal{B}$ and intersects the image of $\mathrm{id} \times F$, *i.e.*, the graph of F, transversely: The latter claim can be verified by calculating differentials. Thus X_w is a smooth variety of dimension $\ell(w)$, where $\ell(w) = \dim (BwB)/B$.

Now suppose that (B, T) is an F-stable Borel pair, and set $U = [B, B]$. Recall that up to a choice of section $W \to N_G(T)$, we can define a scheme $\tilde{X}_w \subseteq G/U$, such that the right T-action on G/U restricts to a T^{wF}-action on $\tilde{X}_w$, and the (free) quotient by T^{wF} defines a finite cover $\pi_w : \tilde{X}_w \to X_w$. We have a commutative square:

$$\begin{array}{ccc} \tilde{X}_w & \longrightarrow & G/U \\ {\scriptstyle \pi_w} \big\downarrow & & \big\downarrow \\ X_w & \longrightarrow & G/B \simeq \mathcal{B} \end{array}$$

We draw the following conclusions:

(1) The map π_w is finite étale. Thus $\tilde{X}_w$ is also a smooth variety of dimension $\ell(w)$.

(2) The compactly-supported cohomology $\mathrm{H}^*_c(\tilde{X}_w)$ forms a graded (G^F, T^{wF})-bimodule. In particular, if we write $V[\theta]$ for the θ-isotypic component of a representation V of T^{wF}, then the $\bar{\mathbf{Q}}_\ell$-vector space

$$\mathbf{R}_{w,\theta} = \mathbf{R}^{G^F}_{T^{wF}}(\theta) := \mathrm{H}^*_c(\tilde{X}_w)[\theta]$$

is a graded representation of G^F for any character $\theta : T^{wF} \to \bar{\mathbf{Q}}_\ell^\times$.

(3) Pushforward defines a map

$$\pi_{w,!} = \pi_{w,*} : \mathrm{H}^*_c(\tilde{X}_w) \to \mathrm{H}^*_c(X_w).$$

With more work, one can show that it factors through an isomorphism $\mathbf{R}^{G^F}_{T^{wF}}(1) = \mathrm{H}^*_c(\tilde{X}_w)^{T^{wF}} \xrightarrow{\sim} \mathrm{H}^*_c(X_w)$.

We refer to the operation $\mathbf{R}^{G^F}_{T^{wF}}$ as *Deligne–Lusztig induction* from T^{wF} to G^F.

4.3.

In their original paper, Deligne–Lusztig focused on the virtual character of G^F defined by

$$R_{w,\theta} = R^{G^F}_{T^{wF}}(\theta) := \sum_i (-1)^i \mathrm{H}^i_c(\tilde{X}_w)[\theta].$$

Indeed this alternating sum resembles that appearing in the Lefschetz formula, which suggests that $R_{w,\theta}$ is related to point-counting, hence more tractable than $\mathbf{R}_{w,\theta}$ itself for general w and θ.

Note that if F acts nontrivially on W, then X_w and $\tilde{X}_w$ need not be stable under the Frobenius maps on G/B and G/U induced by F. Nonetheless, there must be some $\delta \geq 1$ such that F^δ acts trivially on W. By Geck Exercise 4.7.3(a), F^δ is also a Frobenius map on G. (If F corresponds to an $\mathbf{F}_q$-rational structure, then F^δ corresponds to an $\mathbf{F}_{q^\delta}$-rational structure.) Since O_w and the graph of F are both F^δ-stable in $\mathcal{B} \times \mathcal{B}$, we deduce from the first cartesian square above that X_w is F^δ-stable as well.

Whether or not $\tilde{X}_w$ is F^δ-stable depends on how we choose the section $w \mapsto \dot{w} : W \to N_G(T)$. Observe that $W = W^{F^\delta} = N_{G^{F^\delta}}(T^{F^\delta})/T^{F^\delta}$. Thus, for all w, we can choose $\dot{w} \in N_{G^{F^\delta}}(T^{F^\delta})$, and in this case, $\tilde{X}_w$ is F^δ-stable.

4.4.

Take $G = \mathrm{SL}_2$ and F the standard Frobenius, so that we can write $W = \{e, s\}$. Since F acts trivially on W, the varieties X_e and X_s are F-stable.

We saw last time that X_e is a set of $q+1$ points and $X_s = \mathbf{P}^1 \setminus X_e$. In particular, X_s is affine of dimension 1, so we know that $\mathrm{H}^0_c(X_s) = 0$ and the remaining compactly-supported cohomology of X_s is supported in degrees 1 and 2. Similarly, the compactly-supported cohomology of X_e is supported in degree 0, where it is a vector space of dimension $q+1$.

The long exact sequence from the inclusion $j : X_s \to \mathbf{P}^1$ gives

$$\cdots \to 0 = \mathrm{H}^1_c(X_e) \to \mathrm{H}^2_c(X_s) \xrightarrow{j_!} \mathrm{H}^2_c(\mathbf{P}^1) \to \mathrm{H}^2_c(X_e) = 0 \to \cdots$$

from which $\mathrm{H}^2_c(X_s) \simeq \mathrm{H}^2_c(\mathbf{P}^1) \simeq \bar{\mathbf{Q}}_\ell(-1)$, and

$$\cdots \to 0 = \mathrm{H}^0_c(X_s) \xrightarrow{j_!} \mathrm{H}^0_c(\mathbf{P}^1) \to \mathrm{H}^0_c(X_e) \to \mathrm{H}^1_c(X_s) \xrightarrow{j_!} \mathrm{H}^1_c(\mathbf{P}^1) = 0 \to \cdots$$

from which $\mathrm{H}^1_c(X_s) \simeq \mathrm{H}^0_c(X_e)/\mathrm{H}^0_c(\mathbf{P}^1) \simeq \bar{\mathbf{Q}}_\ell^{\oplus q}$. In particular,

$$\mathrm{tr}(F \mid \mathrm{H}^1_c(X_s)) = \mathrm{tr}(F \mid \mathrm{H}^2_c(X_s)) = q.$$

This agrees with the sanity check from Lefschetz: $|X_s^F| = 0$ by construction, matching $0 - q + q = 0$.

Note that $\mathrm{H}^1_c(X_s)$ and $\mathrm{H}^2_c(X_s)$ individually define representations of G^F. With more work, one can show that their respective characters are St, the Steinberg character, and 1, the trivial character, in the notation from the previous set of notes. Unfortunately, this means that $\mathbf{R}_{s,1} = \mathrm{H}^*(X_s)$ fails to see anything new: We have only reproduced the principal series from last time. Even so, we see something interesting on virtual characters:

$$R_{e,1} = 1 + \mathrm{St},$$
$$R_{s,1} = 1 - \mathrm{St},$$

so under the pairing $(-,-)_{G^F}$ on class functions induced by the Hom_{G^F}-pairing on isomorphism classes of representations, we have $(R_{e,1}, R_{s,1})_{G^F} = 1 - 1 = 0$: *i.e.*, $R_{e,1}$ and $R_{s,1}$ are orthogonal.