3.

Throughout, G is a connected, smooth, affine algebraic group over $k = \bar{\mathbf{F}}_q$ with a Frobenius map $F : G \to G$ corresponding to an $\mathbf{F}_q$-form, and (B, T) is an F-stable Borel pair.

Today we explain how Deligne–Lusztig found a generalization of the induction functor $\mathrm{Ind}_{B^F}^{G^F}$ depending on algebraic geometry.

3.1.

As motivation, we work out the role of the principal series in the character table of $\mathrm{SL}_2(\mathbf{F}_3)$. Take $G = \mathrm{SL}_2$ with the standard Frobenius F, so that $G^F = \mathrm{SL}_2(\mathbf{F}_3)$. Take B upper-triangular, T diagonal, and $U = [B, B]$, so that B, T, U are all F-stable. Let $i = \sqrt{-1}$, so that $\mathbf{F}_9 = \mathbf{F}_3[i]$.

3.1.1.

To determine the conjugacy classes of G^F, we first work over k, then descend. By Jordan, the conjugacy classes of $G(k)$ have representatives

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \text{ possibly double-counted by } a \in k^\times, \quad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}.$$

The conjugacy class of $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ intersects G^F if and only if $a + a^{-1} \in \mathbf{F}_3$. Either $a \in \mathbf{F}_3$ or a is quadratic over $\mathbf{F}_3$. In the former case, $a = \pm 1$. In the latter case, $a = x + yi$ for some $x, y \in \mathbf{F}_3$ with $y \neq 0$, and only $a = \pm i$ works.

Next we have to check which of these conjugacy classes, upon restriction to G^F, breaks into smaller conjugacy classes. It turns out that this happens for the classes whose representatives are single Jordan blocks: They break into four classes with representatives

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}.$$

For instance, to show that the first two are not conjugate under G^F, observe that the conjugating matrix would normalize B, so by a theorem from last time, it would belong to B^F, at which point we can check by direct computation.

It turns out that the other conjugacy classes do not break apart upon restriction. To see this, we list out all of the classes we have, compute the orders of their centralizers in G^F, then verify that the reciprocals add up to 1.

g	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$
$Z_{G^F}(g)$	G^F	G^F	$N_{G^F}(T^F)$	B^F	B^F	B^F	B^F
$\lvert Z_{G^F}(g)\rvert$	24	24	4	6	6	6	6

Note that the properties $Z(G^F) = T^F$ and $Z_{G^F}(U^F) = B^F$ are special to this base field. Both fail when we replace $\mathbf{F}_3$ with $\mathbf{F}_q$ for general q, as we will discuss later.

3.1.2.

By general character theory, we deduce that G^F has 7 irreducible characters.

On Problem Set 1, you will show that the summands of the principal series representations $I_\theta = \mathrm{Ind}_{B^F}^{G^F}(\theta)$ contribute 4 of them. The possibilities for the character $\theta : T^F \to \mathbf{C}^\times$ are the trivial character 1 and an order-2 character α. It turns out that $I_1 \simeq 1 \oplus \mathrm{St}$, where 1 is the trivial character and $\mathrm{St} = \mathrm{St}_{G^F}$ is an irreducible known as the *Steinberg character*; and that $I_\alpha \simeq \rho_+(\alpha) \oplus \rho_-(\alpha)$, where $\rho_\pm(\alpha)$ are also irreducibles of G^F. We get this partial character table (Table 11.1 in Bonnafé):

	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$
I_1	4	4	0	1	1	1	1
1	1	1	1	1	1	1	1
St	3	3	-1	0	0	0	0
I_α	4	-4	0	1	1	-1	-1
$\rho_+(\alpha)$	2	-2	0	$-\bar{\omega}$	$-\omega$	$\bar{\omega}$	ω
$\rho_-(\alpha)$	2	-2	0	$-\omega$	$-\bar{\omega}$	ω	$\bar{\omega}$

Above, ω is a primitive cube root of unity and $\bar{\omega} = \omega^2$. (The characters of I_1, I_α are easier to determine than those of their irreducibles.) The point is that three irreducibles are missing.

3.2.

For general G and $\mathbf{F}_q$: The vector space I_θ has commuting actions of G^F and

$$H_\theta = H_{T^F}^{G^F}(\theta) := \mathrm{End}_{G^F}(I_\theta).$$

It turns out that $\mathbf{C}G^F = \mathrm{End}_{H_\theta}(I_\theta)$, so the double centralizer theorem gives an isomorphism of (G^F, H_θ)-bimodules

$$I_\theta \simeq \bigoplus_M \rho_M \otimes M,$$

where M runs over simple H_θ-modules up to isomorphism and $\rho_M \in \mathrm{Irr}\, G^F$ for all M. By Mackey, we have an isomorphism of vector spaces

$$H_\theta \simeq \bigoplus_{w \in W^F} \mathrm{Hom}_{T^F}({}^w\theta, \theta),$$

where $\dim \mathrm{Hom}_{T^F}({}^w\theta, \theta) = |\{w \mid {}^w\theta = \theta \text{ on } T^F\}|$. In particular:

- $I_{\theta^w} \simeq I_\theta$ for all $w \in W^F$.

- H_θ is largest when $\theta = 1$. It turns out that $H_\theta \simeq \mathbf{C}W^F$.
- H_θ is one-dimensional when θ is sufficiently generic.

When $G = \mathrm{SL}_2$ and F is the standard Frobenius, $W^F = W = S_2$. When q is large enough, generic characters predominate. More precisely, it turns out that for q odd, the principal series contribute $2 + 2 + \frac{q-3}{2}$ of the irreducible characters of G^F. By comparison, here are the conjugacy classes of G^F for q odd, from notes of Paul Garrett:

(1) 2 central conjugacy classes.
(2) $\frac{q-3}{2}$ non-central diagonal conjugacy classes.
(3) $\frac{q-1}{2}$ non-diagonal semisimple conjugacy classes.
(4) 4 non-semisimple conjugacy classes.

3.3.

The key idea is to realize that there are other "finite maximal tori" in G^F besides T^F, whose characters we can also use to build representations of G^F.

Going back to $G = \mathrm{SL}_2$ over $k = \bar{\mathbf{F}}_q$ with the standard Frobenius, recall that $G/B = \mathbf{P}^1$. We have

$$I_1 = \{\text{functions on } G^F/B^F\} = \{\text{functions on } (G/B)^F, \textit{i.e.}, \mathbf{P}^1(\mathbf{F}_q)\}.$$

The open complement $G/B \setminus (G/B)^F$, whose k-points form $\mathbf{P}^1(k) \setminus \mathbf{P}^1(\mathbf{F}_q)$, is still stable under left multiplication by G^F. However, it is not clear what sort of function space would give a finite-dimensional representation of G^F.

Drinfeld observed that instead of a vector space of functions, one might use the vector spaces afforded by a cohomology theory. He worked out the story for SL_2 and told his idea to Deligne–Lusztig. Then the latter worked out the story for general G.

To motivate the geometry in the general situation, first observe that

$$\text{for } G = \mathrm{SL}_2 \text{ and } F \text{ standard,} \quad \begin{cases} \mathbf{P}^1(\mathbf{F}_q) = \{gB \mid F(g)B = gB\}, \\ \mathbf{P}^1(k) \setminus \mathbf{P}^1(\mathbf{F}_q) = \{gB \mid F(g)B \neq gB\}. \end{cases}$$

For a general reductive algebraic group G, recall that the Weyl group $W = N_G(T)/T$ is independent of the maximal torus $T \subseteq G$, since all such tori are conjugate. The $G(k)$-orbits on $(G/B \times G/B)(k)$ are indexed by W via

$$G(k)\backslash(G/B \times G/B)(k) \simeq B(k)\backslash G(k)/B(k) \simeq W.$$

We say that (yB, xB) is in *relative position* $w \in W$ if and only if it goes to w under this bijection, meaning $By^{-1}xB = BwB$. In this case we write $yB \xrightarrow{w} xB$. For general reductive G, let $X_w \subseteq G/B$ be the closed subvariety

$$\begin{aligned} X_w &= \{gB \in G/B \mid F(g)B \xrightarrow{w} gB\} \\ &= \{gB \in G/B \mid g^{-1}F(g) \in BwB\}. \end{aligned}$$

Example 3.1. For any G (and choice of F-stable Borel $B \subseteq G$), the identity element $e \in W$ yields $X_e = (G/B)^F = G^F/B^F$.

Example 3.2. For $G = \mathrm{SL}_2$ and F standard, we can write $W = \{e, s\}$. Then $X_s = G/B \setminus (G/B)^F$.

Drinfeld actually introduced a richer construction. Write $U = [B, B]$, so that $B = T \ltimes U$. Recall that here,

$$\bigoplus_{\theta : B^F \to T^F \to \mathbf{C}^\times} I_\theta = \{\text{functions on } G^F/U^F\}.$$

The G-action on G/U from the left commutes with the T-action from the right by $gU \cdot t = gtU$. These actions descend to commuting G^F- and T^F-actions on G^F/U^F. The projection map $G^F/U^F \to G^F/B^F$ is the quotient by T^F.

Fix a choice of section $w \mapsto \dot{w} : W \to N_G(T)/T$. For general $w \in W$, let $\tilde{X}_w \subseteq G/U$ be the closed subvariety

$$\tilde{X}_w = \{gU \in G/U \mid g^{-1}F(g) \in U\dot{w}U\}.$$

This still has a left G^F-action, but not necessarily a right T^F-action. Instead, a new group appears.

To explain: If G is connected, $H \subseteq G$ is F-stable, and $g \in G(k)$ normalizes H, then Geck Exercise 4.7.5 shows that the map $gF : H \to H$ defined by $[gF](h) = gF(h)g^{-1}$ is another Frobenius map on H. (The connectedness of G is only used to apply Lang's theorem to g.) In particular, each element $w \in W$ gives a Frobenius map $wF : T \to T$.

Lemma 3.3. *The finite group T^{wF} acts freely on $\tilde{X}_w$ from the right. The map $\tilde{X}_w \to X_w$ is the quotient by T^{wF}.*

Proof sketch. We prove the first statement at the level of points. If $gU \in \tilde{X}_w$ and $t \in T^{wF}$, then $F(t) = \dot{w}^{-1}t\dot{w}$, so

$$\begin{aligned}(gt)^{-1}F(gt) &\in t^{-1}(U\dot{w}U)F(t)\\ &= t^{-1}(U\dot{w}U)(\dot{w}^{-1}t\dot{w})\\ &= Ut^{-1}\dot{w}(\dot{w}^{-1}t\dot{w})U\\ &= U\dot{w}U.\end{aligned}$$

So the action is well-defined. It is free because the T-action on G/U is free. □

Example 3.4. For any G (and F-stable B with $U = [B, B]$), taking the lift $\dot{e} = 1$ yields $\tilde{X}_e = G^F/U^F$. If we lift e to a different element of T, then we get an isomorphic but different subvariety of G/U.

Example 3.5. On Problem Set 1, you will compute $\tilde{X}_s$ for $G = \mathrm{SL}_2$ and F standard and $\dot{s} = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. It turns out to be a curve inside $G/U \simeq \mathbf{A}^2 \setminus \{0\}$, sometimes called the *Drinfeld curve*.

Remark 3.6. There is a particularly nice section from W to $N_G(T)/T$, introduced by Tits in his paper "Normalisateurs de tores. I. Groupes de Coxeter etendus". We may return to it later.

Remark 3.7. Here is a perspective suggested by Geordie Williamson on MathOverflow.[1] For $G = \mathrm{SL}_2$, the inclusion of $(G/B)^F = \mathbf{P}^1(\mathbf{F}_q)$ into $(G/B)(k) = \mathbf{P}^1(k)$ is analogous to the inclusion of $\mathbf{RP}^1$ into $\mathbf{CP}^1$. In this sense, the G^F-action on X_s is analogous to the $\mathrm{SL}_2(\mathbf{R})$-action on the open upper and lower half-planes in $\mathbf{C}$. Recall that the interaction between the upper half-plane and its real boundary plays an important role in the theory of modular forms, and hence, the representation theory of $\mathrm{SL}_2(\mathbf{R})$ and its subgroups.

Some issues with this analogy: X_s seems to be analogous to a union of two half-planes rather than a single one. Moreover, X_s has no analogue of the homogeneous description of the upper half-plane as $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$.

[1] See `https://mathoverflow.net/a/188658`.