MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #7

SPRING 2025

Due Wednesday, April 2. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 330, #1)**.** Show that if $h, h' : X \to Y$ are homotopic continuous maps, and similarly, $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Problem 2. Let $x_0, x_1 \in X$. Recall that if $\alpha : [0,1] \to X$ is a path from x_0 to x_1, and $\bar{\alpha}(s) = \alpha(1-s)$ is the reverse path, then

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{defined by } \hat{\alpha}([\gamma]) = [\bar{\alpha}] * [\gamma] * [\alpha]$$

is an isomorphism of groups (Munkres Thm. 52.1).

Show that $\hat{\alpha}$ only depends on the path-homotopy class $[\alpha]$. That is, if β is path-homotopic to α, then $\hat{\alpha} = \hat{\beta}$. Later in class, using Problem 3 below, we will show that the converse is false.

Problem 3 (Munkres 335, #3)**.** Let $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if, for every pair of paths α, β from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Problem 4 (Munkres 334, #1)**.** A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if, for some point $a_0 \in A$, any line segment joining a_0 to any other point of A is contained in A.

(1) Give a star convex subset of $\mathbf{R}^2$ that is not convex.
(2) Show that if A is star convex, then A is simply connected.

Problem 5. Classify the following letter shapes up to: (1) homeomorphism; (2) homotopy equivalence.

A, B, C, D, E, F, G, H, I, J.

You are not required to write down explicit homeomorphisms or homotopy equivalences. Nonetheless, provide some informal reasoning for your classification.

Problem 6. Let $X = [0,1] \times [0,1]$ (in its analytic topology). We define the (closed) *Möbius band* to be the quotient space $\mathcal{M} = X/{\sim}$, where $(0, y) \sim (1, 1-y)$ for all y, and no other pairs of distinct points of X get identified under $\sim$.

Recall that the circle S^1 is homeomorphic to a similar quotient space. Give an explicit homotopy equivalence between $\mathcal{M}$ and S^1. (Hence, they have the same fundamental groups.)

Problem 7 (Munkres 366, #5)**.** Let X be any topological space. Show that the identity map of X is homotopic to a constant map if and only if X is homotopy equivalent to a point. (In this case, X is said to be *contractible*.)

Problem 8 (Munkres 370, #4(2))**.** Recall that a group homomorphism is *trivial* if and only if it sends every element of the domain to the identity element of the target. Give an example of a space X, point $x_0 \in X$, and open $U, V \subseteq X$ containing x_0 such that:

- $X = U \cup V$.
- $U \cap V$ is path-connected.
- $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ are both nontrivial.
- The homomorphisms $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ are both trivial.

Justify that each condition holds for your example.