MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #5

SPRING 2025

Due Wednesday, March 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 152, #5)**.** We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.
(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.

Problem 2 (Munkres 157, #1(a), (c))**.** (1) Show that no two of the spaces $(0,1),(0,1],[0,1]$ are homeomorphic. *Hint:* What happens if you remove certain points from each of these spaces?
(2) Show that $\mathbf{R}$ is not homeomorphic to $\mathbf{R}^n$ for any $n > 1$.

Problem 3 (Munkres 158, #3)**.** Let $f : X \to X$ be continuous.

(1) Show that if $X = [0,1]$, then f has a *fixed point*: that is, a point $x \in X$ such that $f(x) = x$.
(2) Show that the analogue of (1) fails for $X = [0,1)$.

Problem 4 (Munkres 162, #2)**.** (1) What are the components and path components of $\mathbf{R}^\omega$ in the product topology?
(2) Give $\mathbf{R}^\omega$ the uniform topology. Show that $x = (x_n)_n$ and $y = (y_n)_n$ belong to the same connected component of $\mathbf{R}^\omega$ if and only if $x - y := (x_i - y_i)_i$ is bounded. *Hint:* Reduce to the case where $y = (0,0,0,\ldots)$.
(3) Give $\mathbf{R}^\omega$ the box topology. Show that x and y belong to the same connected component of $\mathbf{R}^\omega$ if and only if $x - y$ is *eventually zero*: that is, $x_i = y_i$ for all i large enough. *Hint:* Show that if $x - y$ is not eventually zero, then we can construct a self-homeomorphism of $\mathbf{R}^\omega$ such that $h(x)$ is bounded but $h(y)$ is unbounded, and derive a contradiction.

Problem 5 (Munkres 162, #4)**.** Show that if X is locally path connected, then every connected open subset of X is path connected.

Problem 6 (Munkres 171, #5)**.** Let X be Hausdorff, and let A, B be disjoint compact subspaces of X. Show that there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Problem 7 (Munkres 171, #7)**.** Show that if Y is compact, then for any space X, the projection $\mathrm{pr}_1 : X \times Y \to X$ is a closed map.

Problem 8 (Munkres 171, #8)**.** Let Y be compact Hausdorff. Show that a map $f : X \to Y$ is continuous if and only if its *graph*

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is closed in $X \times Y$. *Hint:* Observe that if Γ_f is closed and V is an open neighborhood of $f(x_0)$ for some $x_0 \in X$, then $\Gamma_f \cap (X \times (Y - V))$ is closed. Apply Problem 7.