MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #3

SPRING 2025

Due Wednesday, February 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let $f : X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y : Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y : Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Problem 2 (Munkres 112, #10)**.** Show that if $f : A \to B$ and $g : C \to D$ are continuous maps, then $(f, g) : A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.

Problem 3 (Munkres 128, #4(1))**.** Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$. In which topologies are the following functions continuous?

$$f(t) = (t, 2t, 3t, \ldots), \qquad g(t) = (t, t, t, \ldots), \qquad h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots).$$

Problem 4 (Munkres 128, #4(2))**.** Same setup as Problem 3. In which topologies do the following sequences converge?

$$\begin{aligned} (w_i)_i \text{ where } w_1 &= (1, 1, 1, 1, \ldots), \\ w_2 &= (0, 2, 2, 2, \ldots), \\ w_3 &= (0, 0, 3, 3, \ldots), \\ &\ldots \end{aligned} \qquad \begin{aligned} (x_i)_i \text{ where } x_1 &= (1, 1, 1, 1, \ldots), \\ x_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots) \\ x_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \ldots), \\ &\ldots \end{aligned}$$

$$\begin{aligned} (y_i)_i \text{ where } y_1 &= (1, 0, 0, 0, \ldots), \\ y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), \\ y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots), \\ &\ldots \end{aligned} \qquad \begin{aligned} (z_i)_i \text{ where } z_1 &= (1, 1, 0, 0, \ldots), \\ z_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots) \\ z_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots), \\ &\ldots \end{aligned}$$

Problem 5 (Munkres 118, #7)**.** What is the closure of $\mathbf{R}^\infty$...

(1) ...in the box topology on $\mathbf{R}^\omega$?
(2) ...in the product topology on $\mathbf{R}^\omega$?

(Silently compare (1)–(2) to the answer in the uniform topology, which you computed for Problem Set 2, #8.)

Problem 6 (Munkres 118, #8)**.** Fix sequences $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \mathbf{R}^\omega$ such that $a_i > 0$ for all i. Let $h : \mathbf{R}^\omega \to \mathbf{R}^\omega$ be defined by

$$h(x_1, x_2, \ldots) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$

(1) Show that in the product topology, h is a self-homeomorphism of $\mathbf{R}^\omega$.
(2) What happens in the box topology?

Problem 7 (Munkres 127, #7)**.** Now consider the map h in Problem 6 in the uniform topology on $\mathbf{R}^\omega$. Under what conditions on $(a_i)_i$ and $(b_i)_i$ is h...

(1) ...continuous?
(2) ...a homeomorphism?

Problem 8 (Munkres 127–128, #8(a)–(b))**.** Let $X \subseteq \mathbf{R}^\omega$ be the subset of sequences x such that $\sum_{i>0} x_i^2$ converges. Then

$$d(x, y) = \sqrt{\sum_{i>0} (x_i - y_i)^2}$$

is a metric on X. The corresponding topology is called the *ℓ^2 topology*.

(1) Show that the ℓ^2 topology is intermediate between the box and uniform topologies that X inherits as a subspace of $\mathbf{R}^\omega$.
(2) Show that the box, ℓ^2, uniform, and product topologies that $\mathbf{R}^\infty$ inherits as a subspace of X are pairwise distinct.