MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #1

SPRING 2025

Due Wednesday, January 22. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let X be a set, and let $\mathcal{T}$ be a collection of subsets of X such that if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. Prove that for any $k > 0$ and $U_1, \ldots, U_k \in \mathcal{T}$, we have $U_1 \cap \cdots \cap U_k \in \mathcal{T}$.

Problem 2 (Munkres 83, #3)**.** Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

is a topology on X. Is the collection

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

a topology on X?

Problem 3. We say that $U \subseteq \mathbf{Z}$ is *evenly-spaced* if and only if it is a (possibly empty) union of sets of the form

$$a\mathbf{Z} + b := \{aq + b \mid q \in \mathbf{Z}\}$$

for various $a, b \in \mathbf{Z}$ with $a \neq 0$. Prove that the collection of evenly-spaced sets is a topology on $\mathbf{Z}$. *Hint:* Use Problem 1 to check the axiom about finite intersections more efficiently.

Problem 4. Let $f : X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Problem 5. Endow $\mathbf{R}$ with the analytic topology. Give an example of a continuous map $f : \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* Pick f to be polynomial.

Problem 6. Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Problem 7. Show that the following topological spaces are homeomorphic:

(1) $\mathbf{R}$.
(2) $(0, \infty)$.
(3) $(0, 1)$.

Above, (1) is endowed with the analytic topology; (2) and (3) are endowed with the subspace topology. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 8. Let X be any topological space, and let $A \subseteq X$, endowed with its subspace topology. Prove that if A is open in X, then a subset of A is open in A if and only if it is open in X.