MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #5

SPRING 2025

Due Wednesday, March 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 3/3 at 1 am, in red.**

Problem 1. Let $S : F^3 \to F^3$ be a linear operator. Give examples where...

(1) ... $\ker(S) \cap \operatorname{im}(S) = \{\vec{0}\}$, but S is not a projection.
(2) ... $\ker(S) \cap \operatorname{im}(S) \neq \{\vec{0}\}$, but S is not nilpotent.

Problem 2. Let V be any vector space, and let $A, B : V \to V$ be linear operators such that $A \circ B = B \circ A$.

(1) Show that any eigenspace for A is B-stable.
(2) Suppose that V is finite-dimensional. Using (1), show that if there is a basis of V in which the matrix of A is diagonal with *pairwise distinct* diagonal entries, then it is also a basis in which the matrix of B is diagonal.

Problem 3. Suppose that V is finite-dimensional, and that $v_1, \ldots, v_m \in V$ is a (nonempty) list of vectors. Show that if $v_1, \ldots, v_m$ are eigenvectors of some linear operator on V, with *pairwise distinct* eigenvalues, then $\{v_1, \ldots, v_m\}$ is a linearly independent set. *Hint:* Induction.

Problem 4 (Axler §5A, #19–20)**.** Recall $F^{\mathbf{N}} = \{(z_1, z_2, z_3, \ldots) \mid z_i \in F \text{ for all } i\}$.

(1) Show that the *forward shift operator* T defined by

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

(2) Find all eigenvectors of the *backward shift operator* S defined by

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, z_4, \ldots).$$

Deduce that every element of F occurs as an eigenvalue of S.

Problem 5. Let V be any complex vector space, and let $T : V \to V$ be a linear operator such that $T^n = \mathrm{Id}_V$. Let $\zeta_n = e^{2\pi i/n} \in \mathbf{C}$.

(1) Show that $V = V_0 + V_1 + \cdots + V_{n-1}$, where

$$V_k = \{v \in V \mid Tv = \zeta_n^k v\}.$$

Hint: There are explicit formulas decomposing any $v \in V$ into a sum $v = \sum_{k=0}^{n-1} v^{(k)}$ with $v^{(k)} \in V_k$ for all k. The $n = 2$ case was #4 on Problem Set 4. If you are stuck, start with the $n = 3$ case.

(2) Show that the sum in (1) is a direct-sum decomposition of V.

Problem 6. Fix an integer m, and let $\mathbf{Z}_m$ denote the integers modulo m. The set of $\mathbf{C}$-valued functions on $\mathbf{Z}_m$ forms a complex vector space V under pointwise addition and scaling. Fix a residue $a \in \mathbf{Z}_m$, and let $T_a : V \to V$ be defined by

$$T_a(f)(x) = f(x - a).$$

(1) Show that T_a is a $\mathbf{C}$-linear operator on V that satisfies $T^m = \mathrm{Id}_V$. Hence, by Problem 5, V is a direct sum of eigenspaces for T_a.
(2) Show that each (nonzero) eigenspace of T_1 is one-dimensional. *Hint:* Show that any eigenvector is uniquely determined by its eigenvalue and $f(0)$.
(3) Show that for any $a \in \mathbf{Z}_m$, all (nonzero) eigenspaces of T_a have the same dimension. *Hint:* Consider how T_a affects a basis of eigenvectors for T_1.

Problem 7. Let V be a vector space over F. The *projective space over V* is

$$\mathbf{P}V = \{\text{1-dimensional linear subspaces of } V\}.$$

(This implicitly depends on F, too.) Let $T : V \to V$ be a linear operator.

(1) Show that if T is *invertible*, then there is a well-defined map $\bar{T} : \mathbf{P}V \to \mathbf{P}V$ given by $L \mapsto T(L)$, to be called the *projectivization of T*.
(2) Does $\bar{T} = \mathrm{Id}_{\mathbf{P}V}$ imply that $T = \mathrm{Id}_V$?
(3) Suppose that $F = \mathbf{C}$ and $\dim(V) = 2$ and $T^n = \mathrm{Id}_V$ for some $n > 0$. Show that T is invertible, and that one of two cases must hold: either $\bar{T} = \mathrm{Id}_{\mathbf{P}V}$, or $\bar{T}$ fixes exactly two points. *Hint:* Problem 5.

Problem 8. Look up the definition of the ring of *quaternions* $\mathbf{H}$. It forms a 4-dimensional real vector space with basis $\{1, i, j, k\}$:

$$\mathbf{H} = \{a1 + bi + cj + dk \mid a, b, c, d \in \mathbf{R}\}.$$

(1) Show that $\mathbf{H}$ forms a complex vector space in which the vector addition remains the same, but the scalar multiplication is left multiplication by elements of $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}\} \subseteq \mathbf{H}$.
(2) Determine a basis for $\mathbf{H}$ as a complex vector space.
(3) Let $H \subseteq \mathrm{Mat}_2(\mathbf{C})$ be the complex linear subspace

$$H = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbf{C} \right\}.$$

Using Problem Set 3, #8 as a guide, give an explicit isomorphism of complex vector spaces $M : \mathbf{H} \to H$ such that $M(z_1 z_2) = M(z_1) \cdot M(z_2)$. *Hint:* Reduce this identity the case where z_2 belongs to the basis from (2).