MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #3

SPRING 2025

Due Wednesday, February 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 2/3 at 1:30 pm, in red.**

Problem 1. Let V be a vector space of dimension n. Let E be a set of exactly n vectors. Show that E is linearly independent if and only if E spans V.

Problem 2 (Axler §2C, #10)**.** Fix an integer $m \geq 1$. Let $\mathcal{P}_m$ be the vector space of polynomials over F that are either zero or have degree $\leq m$. Show that $\{p_0, p_1, \ldots, p_m\}$ is a basis of $\mathcal{P}_m$, where

$$p_k(x) = x^k(1-x)^{m-k}.$$

Hint: Since $\dim \mathcal{P}_m = m + 1$, it suffices by Problem 1 to check that they span.

Problem 3. Linear maps of the form $T : V \to V$ are also called *linear operators* on V. Show that if V is finite-dimensional, then

$$V = \ker(T) + \operatorname{im}(T)$$

if and only if $\ker(T) + \operatorname{im}(T)$ is a direct sum.

Problem 4. Let $p \in F[x]$.

(1) Check that multiplication by p is a linear operator $T_p : F[x] \to F[x]$.
(2) Show that $\ker(T_p) + \operatorname{im}(T_p)$ is a direct sum, but if $p \neq 0$ and $p(0) = 0$, then

$$F[x] \neq \ker(T_p) + \operatorname{im}(T_p).$$

Why doesn't this contradict Problem 3?

Problem 5. Find a linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\operatorname{im}(T) \not\subseteq \ker(T)$, but $\operatorname{im}(T \circ T) \subseteq \ker(T)$. What do these conditions imply about $T \circ T$ and $T \circ T \circ T$?

Problem 6. We say that a square matrix M with complex entries is *hermitian* if and only if $M_{i,j} = \bar{M}_{j,i}$ for all i, j, where the bar denotes complex conjugation. Let

$$\mathrm{Herm}_n = \{n \times n \text{ hermitian matrices}\} \subseteq \mathrm{Mat}_n(\mathbf{C}).$$

(1) Show that Herm_n forms an $\mathbf{R}$-linear subspace of $\mathrm{Mat}_n(\mathbf{C})$ under the

addition $(M + M')_{i,j} = M_{i,j} + M'_{i,j}$ and scaling $(\lambda \cdot M)_{i,j} = \lambda M_{i,j}$,

but not a $\mathbf{C}$-linear subspace.
(2) Find a basis for Herm_2 as a vector space over $\mathbf{R}$. What is its dimension?

Problem 7. We say that a square matrix M with complex entries is *skew-hermitian* if and only if $M_{i,j} = -\bar{M}_{j,i}$ for all i, j. Let

$$\text{Skew}_n = \{n \times n \text{ skew-hermitian matrices}\} \subseteq \text{Mat}_n(\mathbf{C}).$$

(1) Show that $\text{Mat}_n(\mathbf{C}) = \text{Herm}_n + \text{Skew}_n$, as real vector spaces, and that the right-hand side is a direct sum.
(2) Using Problem 6, deduce the dimension of Skew_2.

Problem 8. For any complex number $z = a + bi$, let

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \text{Mat}_2(\mathbf{R}).$$

(1) Check that $z = \cos\theta + i\sin\theta$ if and only if (the underlying linear operator of) M_z rotates the standard basis of $\mathbf{R}^2$ in column notation by θ radians counterclockwise.

    Hint: For the "if" direction, observe that $z \mapsto M_z$ is injective.

    (Below, you may take for granted that the linear operator rotates any other vector in $\mathbf{R}^2$ by the same angle.)
(2) Check that $M_{z_1 z_2} = M_{z_1} \cdot M_{z_2}$ for all $z_1, z_2 \in \mathbf{C}$.
(3) Now take $z_j = \cos\theta_j + i\sin\theta_j$ for $j = 1, 2$. Using (1)–(2), deduce the classical formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$.