MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #2

SPRING 2025

Due Wednesday, January 29. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Axler, §1B #8)**.** Let V be a vector space over $\mathbf{R}$. We define its *complexification* to consist of the set $V_{\mathbf{C}} = V^2$ and the operations

$$+ : V_{\mathbf{C}} \times V_{\mathbf{C}} \to V_{\mathbf{C}} \quad \text{and} \quad \cdot : \mathbf{C} \times V_{\mathbf{C}} \to V_{\mathbf{C}}$$

defined thus:

(1) $(u, v) + (u', v') = (u + u', v + v')$.
(2) $(a + bi) \cdot (u, v) = (a \cdot u - b \cdot v, a \cdot v + b \cdot u)$ for all $a, b \in \mathbf{R}$.

Show that the complexification of V forms a vector space over $\mathbf{C}$. *Hint:* It helps to think of (u, v) as a formal linear combination "$u + iv$" even though the actual work is just axiom-checking.

Problem 2. In the setup of Problem 1, suppose that $\{e_\alpha\}_{\alpha \in I}$ is a basis for V. Show that $\{(e_\alpha, 0)\}_{\alpha \in I}$ is a basis for $V_{\mathbf{C}}$.

Problem 3. Let

$$V = \{p \in \mathbf{R}[x] \mid p = 0 \text{ or } \deg(p) \leq 4\},$$

a vector space over $\mathbf{R}$ (that Axler calls $\mathcal{P}_4(\mathbf{R})$). Find a basis of V that contains no polynomials of odd degree (and verify that it is a basis).

Problem 4 (Axler, §2C #3)**.** Keep V as in Problem 3. Let

$$U = \{p \in V \mid p(6) = 0\}.$$

(1) Show that U is a linear subspace of V, and find a basis for U.
(2) Extend the basis in (1) to a basis of V.
(3) Find a linear subspace W of V such that V is the direct sum of U and W.

Problem 5 (Axler, §2C #7)**.** Repeat parts (1)–(3) of Problem 4, but now taking

$$U = \{p \in V \mid \textstyle\int_{-1}^{1} p(x)\, dx = 0\}.$$

Problem 6 (Axler, §2C #15)**.** Suppose that V is a finite-dimensional vector space and that V_1, V_2, V_3 are linear subspaces such that

$$\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V.$$

Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Problem 7 (Axler, §2C #2)**.** Show that the linear subspaces of $\mathbf{R}^3$ are: $\{0\}$, all lines containing the origin, all planes containing the origin, and $\mathbf{R}^3$ itself.

Problem 8. Look up the definition of a *field*. (In Axler §1A, it would be the list of "properties of complex arithmetic" but with a given set $\mathbf{F}$ in place of $\mathbf{C}$.)

(1) Show that the set $\mathbf{F} = \{0, 1\}$ can be given the structure of a field, and that the operations $+$, $\cdot$ are completely determined once we require 0 to be the additive ($+$) identity.
(2) Find all linear subspaces of $\mathbf{F}^3$.