MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #1

SPRING 2025

Due Wednesday, January 22. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Show that if U, W are linear subspaces of a vector space V, then $U \cap W$ is also a linear subspace of V.

Problem 2. Recall that if U, W, W' are linear subspaces of a vector space V, then

$$U \cap (W + W') \text{ and } (U \cap W) + (U \cap W') \text{ need } \textit{not} \text{ be equal.}$$

Show that nonetheless, one is *always* contained in the other.

Problem 3. Let $F \in \{\mathbf{R}, \mathbf{C}\}$. A *magic square* over F is an array of the form

$$M = \boxed{\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}}, \quad \text{with } a, b, c, d, e, f, g, h, i \in F,$$

such that the sums along the three rows, the three columns, and the two main diagonals are equal. Show that the set of magic squares is a vector space over F under the operations

$$\boxed{\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}} + \boxed{\begin{matrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{matrix}} = \boxed{\begin{matrix} a+a' & b+b' & c+c' \\ d+d' & e+e' & f+f' \\ g+g' & h+h' & i+i' \end{matrix}},$$

$$\lambda \cdot \boxed{\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}} = \boxed{\begin{matrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \\ \lambda g & \lambda h & \lambda i \end{matrix}}.$$

Problem 4. Find magic squares $M_1, \ldots, M_k$, where $k < 9$, such that every magic square over F takes the form

$$\lambda_1 M_1 + \cdots + \lambda_k M_k \quad \text{for some } \lambda_1, \ldots, \lambda_k \in F.$$

How small can you get k? (This does not depend on whether $F = \mathbf{R}$ or $F = \mathbf{C}$.)

Problem 5. Let V be a vector space, and let $\{W_\alpha\}_{\alpha \in I}$ be a collection of linear subspaces of V, where I is possibly infinite. We define the *sum* $\sum_\alpha W_\alpha$ to be the set of all elements of V of the form

$$\sum_{\alpha \in J} w_\alpha, \quad \text{where } J \subseteq I \text{ is } \underline{\text{finite}} \text{ and } w_\alpha \in W_\alpha \text{ for all } \alpha.$$

Show that $\sum_\alpha W_\alpha$ is a linear subspace of V.

Problem 6. Let $\mathbf{N}$ be the set of positive integers, and let

$$F^\infty := F^{\mathbf{N}} = \{\text{functions from } \mathbf{N} \text{ into } F\}.$$

For any integer $n > 0$, let

$$V_n = \{\text{functions } f : \mathbf{N} \to F \text{ such that } f(m) = 0 \text{ for all } m > n\},$$
$$W_n = \{\text{functions } f : \mathbf{N} \to F \text{ such that } f(m) = 0 \text{ for all } m \neq n\}.$$

Note that V_n and W_n are linear subspaces of F^∞. Show that

$$\sum_{n>0} W_n = \sum_{n>0} V_n, \quad \text{but that} \quad \sum_{n>0} V_n \neq F^\infty.$$

Problem 7. Recall that $F[x]$ is the set of polynomials in x with coefficients in F. For any integer $n > 0$, let

$$\Gamma_n = \{p(x) \in F[x] \mid \deg(p) < n\}.$$

Note that Γ_n is a linear subspace of $F[x]$. Show that

$$\sum_n \Gamma_n = F[x].$$

Problem 8. Show that any function $f : \mathbf{R} \to \mathbf{R}$ can be decomposed in the form $f = f_0 + f_1$, where

$$f_0(-x) = f_0(x) \quad \text{and} \quad f_1(-x) = -f_1(x),$$

and moreover, that f_0, f_1 are uniquely determined by f. How is this fact related to vector spaces, linear subspaces, sums, and/or direct sums?