MATH 340: ADVANCED LINEAR ALGEBRA
FINAL EXAM GUIDE

SPRING 2025

The midterm exam will be held in **Durham Laboratory (DL) 220, on Tuesday, May 6, at 9:00 am**. It will be a closed-book exam, designed to take < 120 minutes.

What Could Appear.

$\approx$ *Chapter 1–2.*

- definition of F^S for a set S, and of $F[x]$
- what it means for a sum to be a *direct* sum
- how to check that $U + W$ is a direct sum, for subspaces $U, W \subseteq V$
- how to check that a set of vectors is a basis for V
- the formula relating the dimensions of U, W, $U \cap W$, and $U + W$
- definition of $V_{\mathbf{C}}$

$\approx$ *§3A–3D.*

- how kernel and image relate to injectivity and surjectivity
- the formula relating $\dim \ker(T)$, $\dim \operatorname{im}(T)$, and $\dim V$ for a linear map $T : V \to W$
- how to describe linear $T : V \to W$ using (ordered) bases for V and W
- what it means for a linear map to be a linear isomorphism
- how to express the matrix of T in a basis $(e_i)_i$, given the matrix of T in a basis $(v_i)_i$ and the expansions $v_i = \sum_k a_{k,i} e_k$
- definition of conjugacy and commutation for square matrices
- the formal differentiation operator on $F[x]$

$\approx$ *Chapters 5, 8.*

- how to check that a subspace of V is T-stable ($=$ T-invariant), given a linear operator $T : V \to V$
- definitions of eigenlines, eigenvalues, eigenvectors, eigenspaces
- how working over $\mathbf{R}$ versus over $\mathbf{C}$ affects existence of eigenvalues
- distinction between *an* eigenline with eigenvalue λ, *the* eigenspace for λ, and *the* generalized eigenspace for λ
- definitions of diagonalizable, nilpotent, and unipotent operators
- meaning of polynomials evaluated on linear operators (*e.g.*, $(T - \lambda)(T - \mu)$)
- the fundamental theorem of algebra (Axler §4.13)
- how the minimal polynomial is related to eigenvalues
- the statement of the Jordan canonical form theorem
- formulas for the trace, determinant, and characteristic polynomial of a $\mathbf{C}$-linear operator in terms of its eigenvalues
- properties of matrix products under trace and determinant

$\approx$ *§3E–3F, Chapter 9.*

- definitions of $\mathrm{Hom}(V, W)$, $V^\vee$, V/U, $\mathrm{Ann}_{V^\vee}(U)$ and their dimensions[1]
- definition of dual bases and duals of linear operators
- effect of dualizing on injectivity/surjectivity of linear maps between finite-dimensional vector spaces
- relation between vector transpose and dual bases
- relation between matrix transpose and dual linear maps
- how bilinearity is very different from linearity
- examples of bilinear functionals other than the dot product
- definition of $\mathrm{Bil}(W, V)$ and its dimension
- definition of $W \otimes V$ via bilinear functionals and its dimension
- definition of pure tensors and their role in bases
- example of a mixed tensor that is not pure
- definition of multilinear forms and of the alternating property
- $\dim \mathrm{Alt}^n(V)$ when $\dim V = n$
- statement of how matrix determinants relate to alternating forms (Axler §9.40–9.41)

$\approx$ *Chapter 6–7.*

- definitions of skew-linearity and (conjugate) symmetry
- why positive-definiteness is stricter than nondegeneracy for bilinear/skew-linear forms
- definitions of complex inner products and (general) norms
- the norm $\|-\|$ associated with an inner product
- orthogonal projection onto a nonzero vector
- the Cauchy–Schwarz identity (in finite-dimensional vector spaces)
- orthonormal bases and the key ideas in the Gram–Schmidt process
- properties of the orthogonal complement $U^\perp$ to a linear subspace $U \subseteq V$ in an inner product space V
- definition of orthogonal/unitary operators
- rotations in $\mathbf{R}^2$; reflections in arbitrary real inner product spaces
- how to characterize the matrices of orthogonal/unitary operators with respect to an orthonormal basis
- the Riesz representation theorem (Axler §6.42, or my restatement of the conclusion as a linear isomorphism $V \to V^\vee$)
- defining property of the adjoint $T^* : W \to V$ to a linear map $T : V \to W$ between inner product spaces
- relation between self-adjoint operators and symmetric/hermitian matrices
- definition of a normal operator T, and the relation between the eigenspaces of T and T^*
- statement of the spectral theorem
- definition of a Gram matrix and of singular values
- definition of the (L^2) operator norm $\|T\|$ and its relation to singular value

[1] Axler writes V' for our $V^\vee$, and writes U^0 for our $\mathrm{Ann}_{V^\vee}(U)$.

What We'll Have Covered by Then, But Will Not Appear.

- various hard proofs (of Steinitz exchange, the dimension formulas, the Jordan canonical form theorem, the interpretaion of determinant via alternating forms, *etc.*)
- definitions or unusual examples of fields/rings
- $F[\![x]\!]$
- forward/backward shift operators on $F^{\mathbf{N}}$
- projectivization of vector spaces and invertible linear operators
- quaternions
- statement of the Cayley–Hamilton theorem
- the Fibonacci sequence or its closed formula
- relation between the kernel/image of a linear map and its dual
- relation betweeen annihilators and duals of quotients
- $\mathrm{Bil}(W, V \mid U)$
- sign of a permutation
- general isometries between different inner product spaces (Axler §7.44)
- the Cartan–Dieudonné theorem
- the QR and Cholesky decompositions

What We Won't Have Covered by Then.

- the proofs in Axler Chapter 4
- the Gershgorin disk theorem (Axler Thm. 5.67)
- the singular value decomposition
- quadratic forms