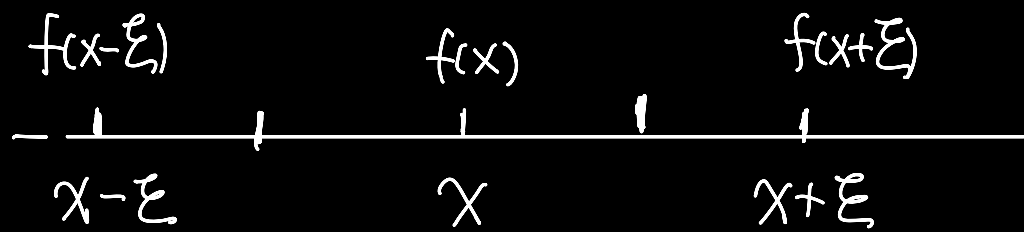


$$f \in C^2(\mathbb{R})$$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$\Delta f(x) = f''(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - 2f(x) + f(x-\varepsilon)}{\varepsilon^2}$$

Fujiwara, 1995

Def. For a directed connected finite graph (V, E) which has property

$$\text{if } [x, y] \in E, \Rightarrow [y, x] \in E$$

a length function: $l : E \to \mathbb{R}_+$

$$l([x, y]) = l([y, x])$$

a weight function $m : V \to \mathbb{R}_+$

$$m(x) = \frac{1}{2} \sum_{x \sim y} l([x, y])$$

Denote $\underline{[x,y] = -[y,x]}$

$$L^2(V) = \{ f : V \to \mathbb{C} \}$$

$$L^2(E) = \{ \phi : E \to \mathbb{C} : \phi(-e) = -\phi(e) \ \forall e \}$$

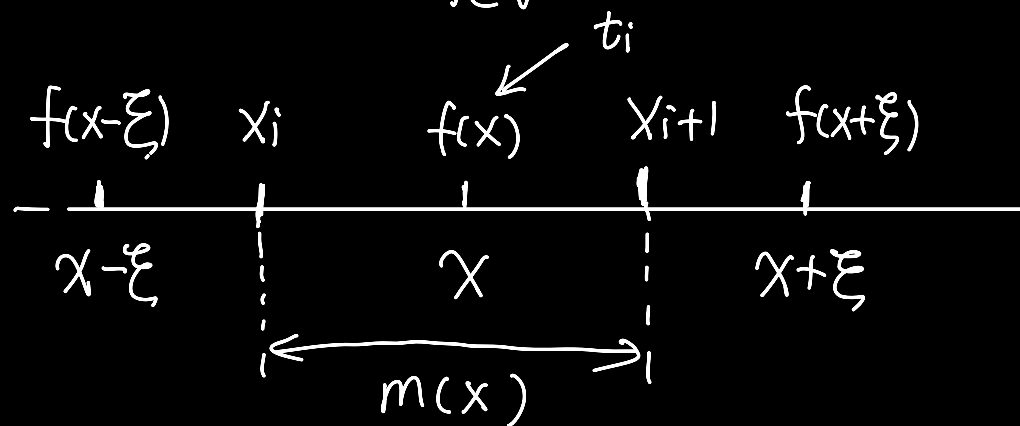
$$\phi([y,x]) = -\phi([x,y])$$

Def. Differential op. $d : L^2(V) \to L^2(E)$

$$df([x,y]) = \frac{f(y) - f(x)}{l([x,y])}$$

Def. Inner products on $L^2(V)$

$$\langle f, g \rangle_V = \sum_{x \in V} m(x)\, f(x)\, \overline{g(x)}$$



$$\int dx\, f(x)\, \bar{g}(x)$$

$$S = \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, f(t_i)$$

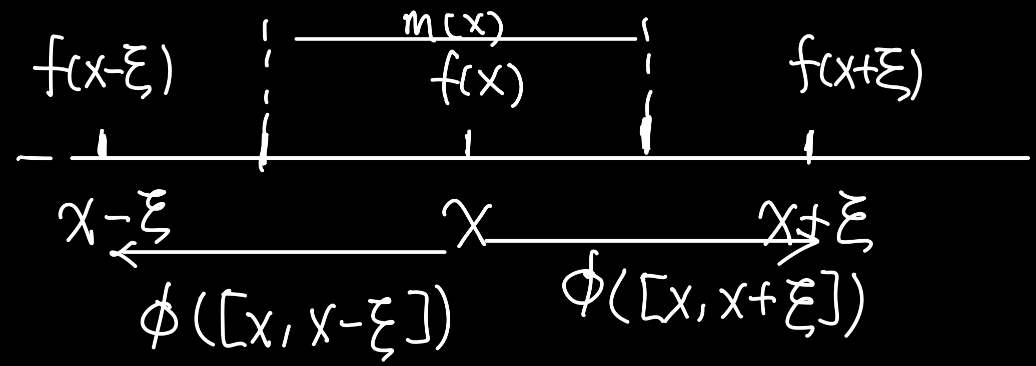
Inner product on $L^2(E)$

$$\langle \phi, \psi \rangle_E \overset{\Delta}{=} \frac{1}{2} \sum_{e \in E} l(e)\, \phi(e) \cdot \overline{\psi}(e)$$

Def. The adjoint op. $\delta : L^2(E) \to L^2(V)$

$$(\delta \phi)(x) = -\frac{1}{m(x)} \sum_{x \sim y} \phi([x, y])$$

ex. $\langle \delta \phi, f \rangle_V = \langle \phi, df \rangle_E$



$$= -\phi([x-\xi, x])$$

$$(\delta\phi)(x) = -\frac{1}{\xi}\left(\phi([x, x+\xi]) - \phi([x-\xi, x])\right)$$

Def. Laplacian $\Delta : L^2(V) \to L^2(V)$

$$\Delta f(x) = -\delta d f(x)$$

$$= \frac{1}{m(x)} \sum_{x \sim y} \frac{f(y) - f(x)}{\ell([x, y])}$$

Prop. ① Δ is a self-adjoint op.

② eigenvalues of Δ are $\mathbb{R}$ & nonpositive

③ Exactly one is 0 $\Rightarrow$ $f(x) = c$

Lemma $\langle \Delta f, f \rangle_V = - \langle df, df \rangle_E$

$$= \langle -\delta df, f \rangle = - \langle df, df \rangle$$

$$\langle \Delta f, f \rangle = - \overline{\langle df, df \rangle}$$

$$=: \overline{\langle \Delta f, f \rangle}$$

$$= \langle f, \Delta f \rangle$$

② If f is an eig. func. of Δ

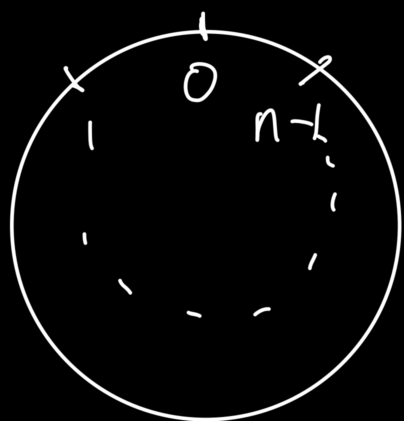
$$\Delta f = \lambda f$$

$$\lambda \langle f, f \rangle = - \langle df, df \rangle$$

$$\lambda = - \frac{\langle df, df \rangle}{\langle f, f \rangle} \leqslant 0$$

③ $df = 0 \underset{\text{connected}}{\Rightarrow} f = c$

Cycle graph $X(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$



$$\ell([j, j+1]) = \frac{1}{n}$$

$$m(j) = \frac{1}{n} \qquad \forall j \in \mathbb{Z}/n\mathbb{Z}$$

$$df([j, j+1]) = n\,(f(j+1) - f(j))$$

$$\begin{aligned} \Delta f(j) &= n^2\,(f(j+1) - f(j) + f(j-1) - f(j)) \\ &= n^2\,(f(j+1) - 2f(j) + f(j-1)) \\ &= \underbrace{\boxed{n^2\,(A - 2I)}}_{= \Delta} f(j) \end{aligned}$$

Recall

Op.	eig. func	eig. value
A	e_a	$2\cos\left(\frac{2\pi a}{n}\right)$
$\Delta = n^2(A-2I)$	e_a	$2n^2\left(\cos\frac{2\pi a}{n} - 1\right) = -4n^2 \cdot \sin^2\frac{\pi a}{n}$
$\frac{d^2}{dx^2} : C^2(\mathbb{R}/\mathbb{Z}) \to \mathbb{C}$	e_a	$-4\pi^2 a^2$

$$-4n^2 \sin^2\frac{\pi a}{n} = -4\pi a^2 \left(1 + O\left(\left(\frac{a}{n}\right)^2\right)\right)$$