MATH 250: TOPOLOGY I PROBLEM SET #3

FALL 2026

Due Monday, November 23. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. Show that no two of the spaces $(0,1), (0,1], [0,1]$ are homeomorphic. *Hint:* What happens when a point is removed from each of these spaces?

Problem 2. Let $f\colon X \to X$ be continuous.

(1) Show that if $X = [0,1]$, then f has a *fixed point*: that is, a point $x \in X$ such that $f(x) = x$.
(2) Show that the analogue of (1) fails for $X = [0,1)$.

Problem 3. (1) What are the components and path components of $\mathbf{R}^\omega$ in the product topology?
(2) Give $\mathbf{R}^\omega$ the uniform topology. Show that $x = (x_n)_n$ and $y = (y_n)_n$ belong to the same connected component of $\mathbf{R}^\omega$ if and only if $x - y := (x_i - y_i)_i$ is bounded. *Hint:* Reduce to the case where $y = (0, 0, 0, \ldots)$.
(3) Give $\mathbf{R}^\omega$ the box topology. Show that x and y belong to the same connected component of $\mathbf{R}^\omega$ if and only if $x - y$ is *eventually zero*: that is, $x_i = y_i$ for all i large enough. *Hint:* Show that if $x - y$ is not eventually zero, then we can construct a self-homeomorphism of $\mathbf{R}^\omega$ such that $h(x)$ is bounded but $h(y)$ is unbounded, and derive a contradiction.

Problem 4. Let X be Hausdorff, and let A, B be disjoint compact subspaces of X. Show that there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Problem 5. Show that if Y is compact, then for any space X, the projection $\mathrm{pr}_1\colon X \times Y \to X$ is a closed map.

Problem 6. Let Y be compact Hausdorff. Show that a map $f\colon X \to Y$ is continuous if and only if its *graph*

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is closed in $X \times Y$. *Hint:* Observe that if Γ_f is closed and V is an open neighborhood of $f(x_0)$ for some $x_0 \in X$, then $\Gamma_f \cap (X \times (Y - V))$ is closed. Apply Problem 5.

Problem 7. Let $f_1, f_2, f_3\colon A \to X$ be continuous maps. Suppose that φ is a homotopy from f_1 to f_2 and ψ is a homotopy from f_2 to f_3. Construct a homotopy from f_1 to f_3 explicitly in terms of φ and ψ.

Problem 8. In the setup of Problem 7, suppose that

$$A = [0,1],$$
$$f_1(0) = f_2(0) = f_3(0),$$
$$f_1(1) = f_2(1) = f_3(1).$$

Show that in this case, if φ and ψ are path homotopies, then we can choose the solution of (1) to be a path homotopy as well.

Problem 9. Let $f, g\colon X \to Y$ and $F, G\colon Y \to Z$ be continuous. Show that

$$\text{if } f \sim g \text{ and } F \sim G, \quad \text{then } F \circ f \sim G \circ g.$$

Problem 10. Use the $f = g$ case of Problem 9 to show: If X, Y are nonempty and Y is contractible, then any continuous map from X into Y is *nulhomotopic*: that is, homotopic to a constant map.

Problem 11. Suppose that:

- $f\colon X \to Y$ and $g\colon Y \to X$ define a homotopy equivalence between X and Y.
- $j\colon Y \to Z$ and $k\colon Z \to Y$ define a homotopy equivalence between Y and Z.

Show that $j \circ f\colon X \to Z$ and $g \circ k\colon Z \to X$ define a homotopy equivalence between X and Z.

Hint: Among other things, you will need to show that $g \circ k \circ j \circ f \sim \mathrm{id}_X$. But we are given that $k \circ j \sim \mathrm{id}_Y$. Use Problem 9 to show that $k \circ j \sim \mathrm{id}_Y$ implies $g \circ k \circ j \circ f \sim \mathrm{id}_X$.

Problem 12. Show that if the identity map of a space X is homotopic to a constant map with value $c \in X$, then X is homotopy equivalent to $\{c\}$. Deduce that all *contractible* spaces are homotopy equivalent to each other.

Problem 13. A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if there is some point $c \in A$ such that the line segment between c and any other point of A is contained within A. That is, for any $a \in A$ and $t \in [0,1]$, we have $(1-t)c + ta \in A$.

(1) Show that if A is star convex, then A is contractible.
(2) Show that if A is star convex, then any loop in A based at a_0 is path homotopic to the constant loop.
(3) Give a star convex subset of $\mathbf{R}^2$ that is not convex.

Problem 14. Let $x_0, x_1 \in X$. Let $\alpha\colon [0,1] \to X$ be a path from x_0 to x_1, and let $\bar{\alpha}$ be the reverse path defined by

$$\bar{\alpha}(s) = \alpha(1-s).$$

(1) Show that $\alpha * \bar{\alpha}$ is path homotopic to e_{x_0}.
(2) Using part (1), show that the map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{given by } \hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is a group homomorphism.

(3) Show that $\hat{\bar{\alpha}}$ is a two-sided inverse of $\hat{\alpha}$. Deduce that $\hat{\alpha}$ is an isomorphism.

Problem 15. Let $f\colon X \to Y$ be a continuous map.

(1) Show that if $\beta, \gamma\colon [0,1] \to X$ are paths such that $\beta(1) = \gamma(0)$, then

$$f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma).$$

(2) Let $x \in X$, Using part (1), show that the map

$$f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x)) \quad \text{given by } f_*([\gamma]) = [f \circ \gamma]$$

is a group homomorphism.

(3) Let $g\colon Y \to Z$ be another continuous map. Show that

$$(g \circ f)_* = g_* \circ f_*$$

as homomorphisms from $\pi_1(X,x)$ into $\pi_1(Z, g(f(x)))$.

Problem 16. Let X be a subspace of some space X', and let $f\colon X \to Y$ be a continuous map. Suppose that

$$f = f'|_X \quad \text{for some continuous map } f'\colon X' \to Y,$$

and that X' is *simply connected*, meaning X' is path connected and its fundamental group is trivial. Show that in this case, the homomorphism $f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ is *trivial* for all $x \in X$, meaning it sends every element of $\pi_1(X,x)$ to the identity element of $\pi_1(Y, f(x))$. *Hint:* Use part (3) of Problem 15.

Problem 17. Let X_1, X_2 be spaces with points $x_1 \in X_1$ and $x_2 \in X_2$. Let $X = X_1 \times X_2$ and $x = (x_1, x_2) \in X$. For $i = 1, 2$, let $\mathrm{pr}_i\colon X \to X_i$ be the appropriate *projection map*..

(1) Show that the maps

$$\varphi\colon \pi_1(X,x) \to \pi_1(X_1, x_1) \times \pi_1(X_2, x_2),$$
$$\psi\colon \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \to \pi_1(X,x)$$

given by

$$\varphi([\gamma]) = ([\mathrm{pr}_1 \circ \gamma], [\mathrm{pr}_2 \circ \gamma]) \quad \text{and} \quad \psi([\gamma_1], [\gamma_2]) = [\gamma_1 \times \gamma_2]$$

are homomorphisms and two-sided inverses of each other. Thus, $\pi_1(X,x) \simeq \pi_1(X_1, x_1) \times \pi_1(X_2, x_2)$.

(2) Using the conclusion to part (2), show that $\mathbf{Z}^n$ is the fundamental group of an explicit (path-connected) topological space, for any integer $n > 0$.

Problem 18. Recall from Problem 13 on the previous problem set that if $s\colon A \to X$ and $r\colon X \to A$ satisfy $r \circ s = \mathrm{id}_A$, then s is called a *section* and r is called a *retraction*.

(1) Show that in this situation,

$$s_*\colon \pi_1(A,a) \to \pi_1(X, s(a)) \text{ is injective for any } a \in A,$$
$$r_*\colon \pi_1(X,x) \to \pi_1(A, r(x)) \text{ is surjective for any } x \in s(A).$$

(2) What if $r \circ s$ is merely a self-homeomorphism of A?

Hint: Use part (3) of Problem 15.

Problem 19. Using Problem 18, show that there is no retraction from the closed disk $\bar{D} := \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ onto its boundary circle S^1. *Hint:* $\bar{D}$ is convex.

Problem 20. Show that:

(1) $\mathbf{R}$ and $\mathbf{R}^n$ are not homeomorphic if $n > 1$. *Hint:* Problem 1.
(2) $\mathbf{R}^2$ and $\mathbf{R}^n$ are not homeomorphic if $n > 2$. *Hint:* Fundamental groups.