MATH 250: TOPOLOGY I PROBLEM SET #2

FALL 2026

Due Wednesday, October 28. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. Let X be arbitrary, and let $d\colon X \times X \to [0,\infty)$ be an arbitrary metric. Let $e\colon X \times X \to [0,\infty)$ be defined by

$$e(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Show that:

(1) e is a bounded metric. *Possible hint:* The mean value theorem shows that if $f(x) = \frac{x}{1+x}$, then $f(a+b) - f(b) \leq f(a)$ for all $a, b \geq 0$.
(2) d and e induce the same topology on X.

Problem 2. We say that metrics $d, e\colon X \times X \to \mathbf{R}$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x,y) \leq Ae(x,y) \text{ and } e(x,y) \leq Bd(x,y) \text{ for all } x, y \in X.$$

In the setting of Problem 1, show that e is <u>not</u> equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 3. Endow $[-1,1]$ with the analytic topology: *i.e.*, the subspace topology it inherits from analytic $\mathbf{R}$. Determine which of the following sets are open in $[-1,1]$, and which are open in $\mathbf{R}$.

$$\begin{aligned}
A &= \{x \mid \tfrac{1}{2} < |x| < 1\},\\
B &= \{x \mid \tfrac{1}{2} < |x| \leq 1\},\\
C &= \{x \mid \tfrac{1}{2} \leq |x| < 1\},\\
D &= \{x \mid \tfrac{1}{2} \leq |x| \leq 1\},\\
E &= \{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbf{Z}_+\}.
\end{aligned}$$

Problem 4. Show that the countable collection

$$\{(a,b) \times (c,d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for the analytic topology on $\mathbf{R}^2$. You may assume one of the problems from the previous problem set. *Hint:* How do the analytic and product topologies on $\mathbf{R}^2$ compare?

Problem 5. The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric*

$$\bar{\rho}(x, y) = \sup_{i>0} \min\{1, |x_i - y_i|\}.$$

For all $x \in \mathbf{R}^\omega$ and $0 < \epsilon < 1$, show that:

(1) The set $U(x, \epsilon) := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots$ is not open in the uniform topology.

(2) Nonetheless, we have

$$B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta<\epsilon} U(x, \delta).$$

Problem 6. Use Problem 5 to verify that the box topology is strictly finer than the uniform topology.

Problem 7. Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. Determine the closure of $\mathbf{R}^\infty$ in the uniform topology on $\mathbf{R}^\omega$.

Problem 8. Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. Determine the closure of $\mathbf{R}^\infty$ in the uniform topology on $\mathbf{R}^\omega$. Silently compare it to the answer for the box and product topologies on $\mathbf{R}^\omega$, which you determined on the previous problem set.

Problem 9. Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$. In which topologies are the following functions from $\mathbf{R}^\omega$ to itself continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots), \\ g(t) &= (t, t, t, \ldots), \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots). \end{aligned}$$

Problem 10. Same setup as Problem 9. In which topologies do the following sequences converge?

$$\begin{aligned} (w_i)_i \text{ where } w_1 &= (1, 1, 1, 1, \ldots), \\ w_2 &= (0, 2, 2, 2, \ldots), \\ w_3 &= (0, 0, 3, 3, \ldots), \\ &\cdots \end{aligned} \qquad \begin{aligned} (x_i)_i \text{ where } x_1 &= (1, 1, 1, 1, \ldots), \\ x_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots) \\ x_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \ldots), \\ &\cdots \end{aligned}$$

$$\begin{aligned} (y_i)_i \text{ where } y_1 &= (1, 0, 0, 0, \ldots), \\ y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), \\ y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots), \\ &\cdots \end{aligned} \qquad \begin{aligned} (z_i)_i \text{ where } z_1 &= (1, 1, 0, 0, \ldots), \\ z_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots) \\ z_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots), \\ &\cdots \end{aligned}$$

Problem 11. Recall the map $\eta\colon \mathbf{R}^\omega \to \mathbf{R}^\omega$ from the previous problem set, given by

$$\eta(x_1, x_2, \ldots) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots).$$

In the uniform topology on $\mathbf{R}^\omega$, what conditions on $(a_i)_i$ and $(b_i)_i$ are necessary and sufficient for η to be...

(1) ...continuous?
(2) ...a homeomorphism?

Problem 12. Let $(X_\alpha)_\alpha$ be an arbitrary collection of topological spaces. Let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of points in $\prod_\alpha X_\alpha$. Thus, each point $x^{(i)}$ has the form

$$x^{(i)} = (x^{(i)}_\alpha)_\alpha, \quad \text{where } x^{(i)}_\alpha \in X_\alpha \text{ for all } \alpha.$$

(1) Show that in the product topology, the sequence $x^{(1)}, x^{(2)}, \ldots$ converges to a point $x = (x_\alpha)_\alpha$ if and only if, for all indices α, the sequence $x^{(1)}_\alpha, x^{(2)}_\alpha, \ldots$ converges to x_α.
(2) What happens if we replace the product topology with the box topology?

Problem 13. Let $s\colon A \to X$ and $r\colon X \to A$ be maps of sets such that $r \circ s$ is the identity map on A.

(1) Show that s must be injective and r must be surjective.
(2) Now suppose that X and A are endowed with topologies with respect to which s and r are both continuous. Show that the topology on A is the quotient topology induced by the surjective map r.

In the situation above, we say that s is a continuous *section* of r, and that r is a continuous *retraction* of s.

Problem 14. Let $X = \mathbf{R}^2 - \{0\}$, endowed with the subspace topology it inherits from analytic $\mathbf{R}^2$. Let

$$S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\},$$

endowed with the subspace topology it inherits from X.

(1) Give a basis for the above topology on S^1. *Hint:* If $\mathcal{B}$ is a basis for the topology on X, then $\{S^1 \cap B \mid B \in \mathcal{B}\}$ is a basis for the topology on S^1.
(2) Give a retraction of the inclusion map from S^1 into X, in the sense of Problem 13. *Hint:* Polar coordinates.

Problem 15. Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).
(3) X is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

Problem 16. Endow $\mathbf{R}$ with the *K-topology*: the topology generated by the basis consisting of the open intervals (a, b) as well as the sets $(a, b) - K$, where

$$K = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$

Let Y be the quotient space obtained from $\mathbf{R}$ by collapsing K to a point, and let $p : \mathbf{R} \to Y$ be the resulting map.

(1) Show that Y is not Hausdorff, but satisfies the *T_1 condition*: For all $x, y \in Y$, we can find an open set containing x but not y.

(2) Show that $(p, p)^{-1}(\Delta_Y)$ is closed in $\mathbf{R} \times \mathbf{R}$. Hence, by Problem 15(3), the product and quotient topologies on $Y \times Y$ must differ.

Problem 17. Let $(A_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

Problem 18. Let X, Y be connected, and let $A \subseteq X$ and $B \subseteq Y$ be proper subsets. Show that $(X \times Y) - (A \times B)$ is a connected subspace of $X \times Y$.

Problem 19. Let $p \colon X \to Y$ be a quotient map. Show that if Y is connected and each subspace $p^{-1}(y) \subseteq X$ is connected, then X is connected.

Problem 20. We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.

(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.