MATH 250: TOPOLOGY I PROBLEM SET #1

FALL 2026

Due Wednesday, September 23. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. Let $f\colon X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Problem 2. Let X be a topological space, and let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is also open.

Problem 3. Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

always forms a topology on X. Does

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

always form a topology on X?

Problem 4. Suppose that $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets, and the largest topology that is contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset.

Problem 5. Using Munkres Lemma 13.2, show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$$

forms a basis for the analytic topology on $\mathbf{R}$.

Problem 6. Let $a\mathbf{Z} + b = \{aq + b \mid q \in \mathbf{Z}\}$, for any integers a and b. Let

$$\mathcal{B} = \{a\mathbf{Z} + b \mid a, b \in \mathbf{Z} \text{ with } a \neq 0\}.$$

Show that $\mathcal{B}$ forms a basis for some topology on $\mathbf{Z}$. We refer to the topology it generates as the *evenly-spaced topology*. *Hint:* If $x \in a\mathbf{Z} + b$, then $a\mathbf{Z} + b = a\mathbf{Z} + x$.

Problem 7. Endow $\mathbf{Z}$ with the evenly-spaced topology from Problem 6.

(1) Show that if there were finitely many (positive) primes, then $\{1, -1\}$ would be open. You may assume that any integer greater than 1 has a prime divisor.
(2) Deduce that there must be infinitely many primes.

Problem 8. Let X be a topological space, and let $A \subseteq X$ be a subset endowed with the subspace topology.

(1) Give an example where a subset of A is open in A, but not open in X.
(2) Now suppose that <u>A itself is open in X</u>. Prove that a subset of A is open in A if and only if it is open in X.

Problem 9. Endow $\mathbf{R}$ with the analytic topology, and

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \dots\} \cup \{0\}$$

with the subspace topology.

(1) Show that for all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.
(2) Show that $\{0\}$ is closed but not open.

Problem 10. Endow $\mathbf{R}^2$ with the analytic topology. How is the subspace topology on $\mathbf{R}$, viewed as the x-axis of $\mathbf{R}^2$, related to the analytic topology on $\mathbf{R}$?

Problem 11. Endow $\mathbf{R}$ with the analytic topology. Give an example of a continuous, <u>non-constant</u> map $f\colon \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* Pick f to be polynomial.

Problem 12. Show that the following topological spaces are homeomorphic:

(1) $\mathbf{R}$.
(2) $(0, \infty)$.
(3) $(0, 1)$.

Above, (1) is endowed with the analytic topology; (2) and (3) are endowed with the subspace topology. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 13. Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Problem 14. Let $f\colon X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)}\colon f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y\colon Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y\colon Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Problem 15. Show that if $f\colon A \to B$ and $g\colon C \to D$ are continuous maps, then $(f, g)\colon A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.

Problem 16. The *box topology* on $\mathbf{R}^\omega$ is defined as follows: U is open if and only if, for all $x \in U$, there is some set of the form $V = (a_1, b_1) \times (a_2, b_2) \times \cdots$ such that $x \in V \subseteq U$. Show that the box topology really is a topology on $\mathbf{R}^\omega$. (Assume the Axiom of Choice.)

Problem 17. What is the closure of $\mathbf{R}^\infty$...

(1) ...in the box topology on $\mathbf{R}^\omega$?
(2) ...in the product topology on $\mathbf{R}^\omega$?

Problem 18. Fix sequences $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \mathbf{R}^\omega$ such that $a_i > 0$ for all i. Let $\eta\colon \mathbf{R}^\omega \to \mathbf{R}^\omega$ be defined by

$$\eta(x_1, x_2, \ldots) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$

(1) Show that in the product topology, η is a self-homeomorphism of $\mathbf{R}^\omega$.
(2) What happens in the box topology?

Problem 19. Recall the notion of a *group* from the initial reading. Show that:

(1) $\mathbf{R}$ forms a group under the law of addition.
(2) $\mathbf{R}$ <u>does not</u> form a group under the law of multiplication.
(3) The set of positive real numbers $\mathbf{R}_+$ forms a group under multiplication.
(4) The set of positive integers $\mathbf{Z}_+$ <u>does not</u> form a group under multiplication.

Problem 20. For part (3), recall or look up the notion of a *subgroup*.

(1) Show that for any set X, the set of bijections from X to itself forms a group under the law of composition (*i.e.*, $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$). This group is usually denoted $\mathrm{Sym}(X)$.
(2) Give two elements $f, g \in \mathrm{Sym}(\{a, b, c\})$ such that $g \circ f \neq f \circ g$.
(3) Suppose that X is endowed with a topology. Show that the set of homeomorphisms from X to itself forms a subgroup of $\mathrm{Sym}(X)$. This subgroup is usually denoted $\mathrm{Homeo}(X)$.