MATH 250: TOPOLOGY I BONUS PROBLEMS

FALL 2026

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Unless otherwise specified, $\mathbf{R}^n$ has the analytic topology for all n, any subset of $\mathbf{R}^n$ has the subspace topology, and any product of topological spaces has the product topology. We write S^{n-1} for the Euclidean unit $(n-1)$-sphere in $\mathbf{R}^n$. For convenience, we write p_θ to denote the point $(\cos 2\pi\theta, \sin 2\pi\theta) \in S^1$.

1. Due Wednesday, September 23

Problem 1. Recall that an *embedding* is an injective continuous map. Give an explicit embedding of the infinite cylinder $S^1 \times \mathbf{R}$ into $\mathbf{R}^2$.

Problem 2. Fix integers a, b. Let $K_{a,b}$ be the set of pairs of complex numbers $(z, w) \in \mathbf{C}^2$ such that $|z| = |w| = 1$ and

$$w^b = z^a.$$

Explain how $K_{a,b}$ can be used to construct an explicit embedding of S^1 into $S^1 \times S^1$ when a and b are coprime. What happens when a and b are not coprime?

Problem 3. Let G be the set of 2×2 complex matrices g such that $\bar{g}^t g = I$, where I is the identity matrix and $\bar{g}^t$ is the complex conjugate of the transpose of g.

(1) Show that G forms a group.
(2) Identifying $\mathbf{C}$ with $\mathbf{R}^2$, show that G is homeomorphic to S^3.
(3) What is the usual name for G in the literature?

Problem 4. An *inverse system* consists of:

- A poset I (say, with partial order $\preceq$).
- A collection of sets $\{X_i\}_{i \in I}$.
- A collection of maps $\{\phi_{i,j} \colon X_j \to X_i\}_{i \preceq j}$, such that for all $i, j, k \in I$ with $i \preceq j \preceq k$, we have $\phi_{i,k} = \phi_{i,j} \circ \phi_{j,k}$.

Show that the following data form an inverse system:

- The poset is the collection I of bounded sets in $\mathbf{R}$ under $\subseteq$.
- The sets are $X_A = \{\text{functions from } A \text{ to } \mathbf{R}\}$ for $A \in I$.
- The map $\phi_{A,B} \colon X_B \to X_A$ is given by $\phi_{A,B}(f) = f|_A$, where $|_A$ means restriction of domain, for all $A, B \in \mathcal{B}$ with $A \subseteq B$.

Problem 5. Given an inverse system with sets $\{X_i\}_{i\in I}$ and maps $\{\phi_{i,j}\colon X_j \to X_i\}_{i\preceq j}$, as above, we define the *inverse limit* $\varprojlim_i X_i$ to be the set

$$\varprojlim_i X_i = \left\{ (x_i)_i \in \prod_{i\in I} X_i \;\middle|\; \phi_{i,j}(x_j) = x_i \text{ for all } i,j \in I \text{ with } i \preceq j \right\}.$$

Let $X_{\mathbf{R}} = \{\text{functions from } \mathbf{R} \text{ to } \mathbf{R}\}$.

(1) Show that for the inverse system in Problem 4, the map $X_{\mathbf{R}} \to \varprojlim_A X_A$ defined by $f \mapsto \{f|_A\}_{A\in I}$ is a bijection.
(2) If we replace "function" with "bounded function" in the definitions of X_A and $X_{\mathbf{R}}$, does (1) still hold?

Problem 6. Fix a commutative ring R and an ideal $N \subseteq R$.

(1) Show that the following data form an inverse system:
   - The poset is $\mathbf{Z}_{>0}$ under $\leq$.
   - The sets are R/N^i for $i \in \mathbf{Z}_{>0}$.
   - The map $\phi_{i,j}\colon R/N^j \to R/N^i$ is the quotient modulo N^i for all $i, j \in \mathbf{Z}_{>0}$ with $i \leq j$.
(2) The inverse limit $\varprojlim_i R/N^i$ is called the *completion* of R with respect to N. Show that it forms a subring of $\prod_i R/N^i$. What is its additive identity 0? What is its multiplicative identity 1?
(3) Show that the diagonal map $R \to \prod_i R/N^i$ factors through a map

$$R \to \varprojlim_i R/N^i.$$

When is the latter injective?

Problem 7. Take $R = \mathbf{Z}$ and $N = p\mathbf{Z}$ for a positive integer p in Problem 6. Here, the inverse limit is called the *ring of p-adic integers* and denoted $\mathbf{Z}_p$.

For any nonzero $\alpha = (\alpha_i)_i \in \mathbf{Z}_p$, let the largest index $i \geq 0$ such that $\alpha_j = 0$ for all $j < i$ be denoted $v(\alpha)$. Let $|0| = 0$, and for nonzero α, let

$$|\alpha| = p^{-v(\alpha)}.$$

(1) Show that $d\colon \mathbf{Z}_p \times \mathbf{Z}_p \to \mathbf{R}$ defined by

$$d(\alpha, \beta) = |\beta - \alpha|$$

is a metric. It is called the *p-adic metric*.
(2) Show that here, the map in Problem 6(3) is injective, so $\mathbf{Z}$ forms a subring of $\mathbf{Z}_p$. But in the p-adic metric, $\mathbf{Z}_p$ is bounded, so the Archimedean property fails.
(3) In the p-adic metric, is $\mathbf{Z}$ dense in $\mathbf{Z}_p$?

2. Due Wednesday, October 28

Problem 8. Let $(\{X_i\}_{i\in I}, \{\phi_{i,j}\}_{i\preceq j})$ be an inverse system. Suppose that each set X_i is endowed with a topology, such that each map $\phi_{i,j}$ is continuous. View $\varprojlim_i X_i$ as a subspace of $\prod_i X_i$ in the product topology. Show that:

(1) If X_i is Hausdorff for all i, then $\varprojlim_i X_i$ is Hausdorff.
(2) If X_i is Hausdorff for all i, then $\varprojlim_i X_i$ is closed in $\prod_i X_i$. *Hint:* Observe that the composition

$$\prod_i X_i \xrightarrow{\mathrm{pr}_j \times \mathrm{pr}_i} X_j \times X_i \xrightarrow{\phi_{i,j} \times \mathrm{id}} X_i \times X_i$$

is continuous for all $i, j \in I$ with $i \preceq j$. Use Problem Set 3, #7(3).
(3) If X_i is compact for all i, then $\varprojlim_i X_i$ is compact. *Hint:* Combine part (2) above with Tychonoff's theorem.

Problem 9. In the setting of Problem 8, show that if each X_i is discrete, then $\varprojlim_i X_i$ is totally disconnected.

Problem 10. A *topological group* is a group G equipped with a topology in which

$$\begin{array}{lll} & \text{the group law } \mu_G\colon G \times G \to G & \text{defined by } \mu_G(g,h) = gh \\ \text{and} & \text{inversion } \iota_G\colon G \to G & \text{defined by } \iota_G(g) = g^{-1} \end{array}$$

are continuous. Show that $\mathbf{R}$ forms a topological group with respect to the addition law and the analytic topology.

Problem 11. Let G be a topological group. Show that:

(1) Any subgroup of G is a topological group in its subspace topology.
(2) Any quotient group of G is a topological group in its quotient topology.

You may assume that for a quotient $p\colon G \to Q$, the product topology on $Q \times Q$ matches the quotient topology it gets from $p \times p$.

Hint: For continuity of μ_Q, we want to show that if $U \subseteq Q$ is open and $V = \mu_Q^{-1}(U) \subseteq Q \times Q$, then V is also open. Next, V is open if and only if $(p \times p)^{-1}(V) \subseteq G \times G$ is open. Finally, observe that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "p\times p"'] & G \arrow[d, "p"] \\ Q \times Q \arrow[r, "\mu_Q"] & Q \end{tikzcd}$$

is *commutative*: that is, $\mu_Q \circ (p \times p) = p \circ \mu_G$.

Problem 12. Let $\{G_i\}_{i\in I}$ be a collection of topological groups.

(1) Show that $\prod_i G_i$ forms a topological group with respect to the coordinate-wise group law and the product topology. *Hint:* Munkres Theorem 19.6.
(2) Now suppose that the G_i fit into an inverse system $(\{G_i\}_{i\in I}, \{\phi_{i,j}\}_{i\preceq j})$, where the $\phi_{i,j}$ are continuous homomorphisms. Deduce from (1) and Problem 11(1) that $\varprojlim_i G_i$ forms a topological group as well.

Problem 13. Recall the set of p-adic integers $\mathbf{Z}_p$ from Problem 7.

(1) Use Problem 12 to show that $\mathbf{Z}_p$ forms a topological group with respect to addition.

(2) Show that $p^j\mathbf{Z}_p$ is a clopen proper subgroup of $\mathbf{Z}_p$ for all integers $j \geq 1$.

(3) Show that by contrast, $\mathbf{R}$ contains no open proper subgroups.

Hint: Using cosets, show that any open subgroup of any topological group is clopen. But $\mathbf{R}$ is connected.

(2)–(3) suggest the remarkable topological differences between p-adic numbers and real numbers.