# MATH 250: TOPOLOGY I PROBLEM SET #6

FALL 2025

**Due Wednesday, November 26.** Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

**Problem 1.** Let $X$ be any space, $x \in X$ any point, and $\gamma : [0,1] \to X$ any path in $X$ starting at $x$. Recall that the *reverse* path $\bar{\gamma}$ is given by $\bar{\gamma}(s) = \gamma(1-s)$. Give an explicit path homotopy from the constant path $e_x$ to $\gamma * \bar{\gamma}$.

(The fact that $e_x$ is path-homotopic to $\gamma * \bar{\gamma}$ is part of Munkres Theorem 51.2, but his proof is indirect.)

*Hint:* Build a path homotopy $h$ such that, for any $t$, the path $h(-,t)$ runs from $x$ to $\gamma(t)$, then runs backward to $x$.

**Problem 2.** Show that if $X$ is path-connected and simply-connected, then for any two points $x, y \in X$, there is a unique path-homotopy class of paths from $x$ to $y$. *Hint:* Given paths $\gamma_0, \gamma_1$ from $x$ to $y$, consider $\gamma_0 * \bar{\gamma}_1 * \gamma_1$.

**Problem 3.** By considering what happens when a point is removed, show that:

(1) $\mathbf{R}^1$ and $\mathbf{R}^n$ are not homeomorphic if $n > 1$.
(2) $\mathbf{R}^2$ and $\mathbf{R}^n$ are not homeomorphic if $n > 2$.

*Hint:* Use different topological invariants in (1) and (2).

**Problem 4.** For each of the following spaces, the fundamental group is either trivial, $\mathbf{Z}$, or $\mathbf{Z} * \mathbf{Z}$. Determine, for each space, which option is the case. You do not need to give explicit homeomorphisms or homotopy equivalences, but give informal descriptions (or pictures) to support your reasoning.

(1) The solid torus $D^2 \times S^1$, where $D^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$.
(2) The hollow torus $S^1 \times S^1$.
(3) The punctured hollow torus $S^1 \times S^1 - \{p\}$, where $p$ is any point.
(4) The cylinder $S^1 \times [0,1]$.
(5) The infinite cylinder $S^1 \times \mathbf{R}$.
(6) $\mathbf{R}^3$ with the nonnegative portions of the $x$, $y$, and $z$-axes deleted.

**Problem 5.** Suppose that $X = U \cap V$, where $U, V$ are open, path-connected, and intersect in a nonempty, path-connected subspace $A$. Give examples where:

(1) $X, U, V$ are simply-connected, but $A$ is not.
(2) $X$ and $U$ are simply-connected, but $V$ is not.
(3) $X$ is simply-connected, but $U$ and $V$ are not.
(4) $A$ is simply-connected, but $X, U, V$ are not.

**Problem 6.** Suppose that $X = X_1 \cup X_2$, where $X_1, X_2$ are closed, path-connected, and intersect in a single point $p$. (Recall that this is an example of a *wedge sum*.) Suppose that for $i = 1, 2$, the subspace $X_i$ contains an open neighborhood of $p$, say $W_i$, such that $\{p\}$ is a deformation retract of $W_i$. Use Seifert–Van Kampen to show that in this situation,

$$\pi_1(X, p) \simeq \pi_1(X_1, p) * \pi_1(X_2, p),$$

as stated in class.

*Hint:* Set $U_1 = X_1 \cup W_2$ and $U_2 = X_2 \cup W_1$. Check that $X_i$ is homotopy equivalent to $U_i$ for $i = 1, 2$, and that $\{p\}$ is homotopy equivalent to $U_1 \cap U_2$. Then check the hypotheses needed for Seifert–Van Kampen. You may assume without proof that path-connectedness is preserved by homotopy equivalence.

**Problem 7** (Munkres 341, #3)**.** Let $p : E \to B$ be a covering map. Show that if $B$ is connected and $p^{-1}(b_0)$ has $k$ elements for some $b_0 \in B$, then $p^{-1}(b)$ has $k$ elements for all $b \in B$. In this case, we say that $p$ is a *$k$-fold covering*.

**Problem 8.** Draw every possible 2-fold covering space of the figure-eight $S^1 \vee S^1$ up to homeomorphism. You do not need to prove that your list is exhaustive.

(Note that $S^1 \vee S^1$ has a symmetry of order two. If two covering spaces differ by a lift of this symmetry, you do not need to draw both.)

**Problem 9** (Munkres 348, #4)**.** Let $\mathbf{R}_+ \subset \mathbf{R}$ be the subset of positive numbers, and let $\mathbf{0} = (0, 0) \in \mathbf{R}^2$. Let $p : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}^2 - \{\mathbf{0}\}$ be defined by

$$p(u, r) = (r \cos(2\pi u), r \sin(2\pi u)).$$

This is a covering map. Find liftings along $p$ of the following paths in $\mathbf{R}^2 - \{\mathbf{0}\}$:

$$\begin{aligned} f(t) &= (2 - t, 0), \\ g(t) &= ((1 + t) \cos(2\pi t), (1 + t) \sin(2\pi t)), \\ h &= f * g. \end{aligned}$$

Sketch (the images of) these paths and their liftings.

**Problem 10.** Let $p : \mathbf{R} \to S^1$ be defined by

$$p(t) = (\cos(2\pi u), \sin(2\pi u)).$$

Consider the path in $S^1 \times S^1$ given by

$$f(t) = ((\cos(2\pi t), \sin(2\pi t)), (\cos(4\pi t), \sin(4\pi t))).$$

Find an explicit lifting $\tilde{f}$ of $f$ along $(p, p) : \mathbf{R} \times \mathbf{R} \to S^1 \times S^1$, and sketch it.