MATH 250: TOPOLOGY I PROBLEM SET #2

FALL 2025

Due Wednesday, September 17. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Last update: 9/16**.

Problem 1. Endow $\mathbf{Z}$ with the evenly-spaced topology from Problem Set 1.

(1) Show that if there were finitely many (positive) primes, then $\{1, -1\}$ would be open. You may assume that any integer greater than 1 has a prime divisor.
(2) Deduce that there must be infinitely many primes.

Problem 2 (Munkres 91–92, #1)**.** Let X be a topological space, and let $A \subseteq Y \subseteq X$ be subsets. Endow Y with the subspace topology that it inherits from X. Show that the subspace topology that A inherits from Y is the subspace topology that A inherits from X.

Problem 3. Let X be a topological space, and let $A \subseteq X$ be a subset endowed with the subspace topology.

(1) Give an example where a subset of A is open in A, but not open in X.
(2) Now suppose that A itself is open in X. Prove that a subset of A is open in A if and only if it is open in X.

Problem 4. Endow $\mathbf{R}$ with the analytic topology, and endow

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology. Show that:

(1) For all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.
(2) $\{0\}$ is closed but not open.

Problem 5. Show that the following topological spaces are homeomorphic:

$$\mathbf{R}, \qquad (0, \infty), \qquad (0, 1).$$

Above, $\mathbf{R}$ is endowed with the analytic topology; $(0, \infty)$ and $(0, 1)$ are endowed with their subspace topologies. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 6. Let $f : X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y : Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y : Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Problem 7 (Munkres 112, #10)**.** Show that if $f : A \to B$ and $g : C \to D$ are continuous maps, then $(f, g) : A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.

Problem 8 (Munkres 128, #9(c)–(d))**.** Recall that the Euclidean norm on $\mathbf{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$, where

$$u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for general $u, v \in \mathbf{R}^n$.

Use the Cauchy–Schwarz inequality $|u \cdot v| \leq \|u\|\|v\|$ to show that $\|u+v\| \leq \|u\|+\|v\|$ for all $u, v \in \mathbf{R}^n$. Conclude that the *Euclidean metric* defined by $d(x, y) = \|x - y\|$ really is a metric on $\mathbf{R}^n$.

Problem 9. Let X be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Assume that the function $e : X \times X \to [0, \infty)$ defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

is a bounded metric. Show that d and e induce the same topology on X.

Note: Munkres 129, #11 asks the reader to prove that e itself is a metric, which is harder.

Problem 10. Let us say that metrics $d, d' : X \times X \to [0, \infty)$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x, y) \leq Ad'(x, y) \text{ and } d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X.$$

In the setting of Problem 9, show that e is not equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 11 (Munkres 127, #6)**.** The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric*

$$\bar{\rho}(x, y) = \sup_{i>0} \min\{1, |x_i - y_i|\}.$$

For all $x \in \mathbf{R}^\omega$ and $0 < \epsilon < 1$, show that:

(1) The following set is not open in the uniform topology:

$$U(x, \epsilon) := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \ldots$$

(2) Nonetheless, $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta<\epsilon} U(x, \delta)$.

Problem 12. The *box topology* on $\mathbf{R}^\omega$ is generated by the basis

$$\{U_1 \times U_2 \times \cdots \mid U_1, U_2, \ldots \text{ are open in } \mathbf{R}\}.$$

(1) Show that the above collection of subsets of $\mathbf{R}^\omega$ really is a basis.
(2) Show that the box topology is strictly finer than the uniform topology. *Hint:* Use Problem 11.