MATH 250: TOPOLOGY I MIDTERM GUIDE

FALL 2025

The midterm exam will be held in-class on **Wednesday, October 8, 2025**. It will start at 12:30 pm and end at 2:00 pm.

You will be allowed to look at any notes on paper that you wrote prior to the exam, and at the textbook (Munkres, *Topology*, 2nd Ed.). However, you will not be allowed to use electronic devices of any kind—including phones, computers, tablets, or other visual/audio devices—or any software.

What Could Appear

§12.

- definition of a topology
- definitions of the discrete, indiscrete, and finite-complement topologies on any set, and why these are topologies
- definition of the analytic topology on $\mathbf{R}^n$
- which of the above topologies on $\mathbf{R}$ are finer or coarser than others

§13.

- what it means for a collection of subsets of X to be a basis
- what it means for a basis to generate a given topology on X
- examples where different bases generate the same topology on X

§18.

- what it means for a map between topological spaces to be continuous
- how to check continuity of $f\colon X \to Y$ using a basis for the topology on Y
- what it means for topological spaces to be homeomorphic
- examples of homeomorphisms between distinct sets (*e.g.*, $\mathbf{R}$ and $(0,1)$)
- examples of continuous bijections that are not homeomorphisms

§16.

- definition of the subspace topology on $A \subseteq X$, given a topology on X
- how the subspace topology on A is related to continuity of the inclusion map from A into X
- examples where some subset of A is open in A, but not in X

§15, 19.

- definition of a direct product of sets $\prod_{i \in I} X_i$
- why $\mathbf{R}^n$ and $\mathbf{R}^\omega$ are examples of direct products of sets
- definitions of the box and product topologies on $\prod_{i \in I} X_i$, given a topology on X_i for each i

- which of the box or product topologies is finer than the other
- how the product topology on $\prod_{i \in I} X_i$ is related to continuity of the various projection maps $\mathrm{pr}_j \colon \prod_{i \in I} X_i \to X_j$
- the closures of $\mathbf{R}^\infty$ in the box and product topologies on $\mathbf{R}^\omega$ (PS 3, #3)

§20.

- definition of a metric
- definition of d-balls for a metric d, and of the topology they generate
- definitions of the euclidean and square metrics on $\mathbf{R}^n$ and on $\mathbf{R}^\infty$
- examples of different metrics on $\mathbf{R}^n$ that generate the same topology
- definition of the uniform topology on $\mathbf{R}^\omega$ (PS 2, #11), and how it compares to the box topology (#12)

§17.

- definitions of the interior and closure of a subset A of a topological space X
- how to check that a point in X belongs to the closure of a subset A
- what it means for a sequence of points to converge to a given point
- definition of the Hausdorff property
- what the Hausdorff property implies for convergence of sequences

§22.

- definition of the quotient topology on a set Q, given a topology on X and a surjective map $f \colon X \to Q$
- how the quotient topology on Q is related to continuity of f
- examples of quotient spaces (*e.g.*, constructed using equivalence relations)

§23–25.

- what it means for a topological space X to be connected, or for subsets $U, V \subseteq X$ to form a separation of X
- how connected subspaces of X interact with separations of X
- how connectedness interacts with continuous maps and finite products
- the fact that (analytic) $\mathbf{R}$ is connected
- why $\mathbf{Q}$ is totally disconnected as a subspace of (analytic) $\mathbf{R}$, but not discrete
- definition of path-connectedness
- definitions of connected components and path components
- what it means to be locally connected or locally path connected
- examples of spaces that are disconnected but locally connected

$\approx$ **§26–27.**

- open-covering definition of compactness
- the fact that (analytic) $[0, 1]$ is compact
- how compactness interacts with continuous maps and finite products

What We'll Have Covered by Then, But Will Not Appear

- the evenly-spaced topology on $\mathbf{Z}$
- the countable-complement topology
- the axiom of choice
- equivalence/inequivalence of metrics (Problem Set 2, #10)
- convergence in the uniform topology on $\mathbf{R}^\omega$
- the intermediate value theorem
- the topologist's sine curve

Practice Problems

Try to hand-write a solution to each problem within 10–15 minutes. Throughout, $\mathbf{R}$ *has the analytic topology unless otherwise specified.*

On the exam, you will not need to use complete sentences, but the clearer your work, the more points you will earn.

Problem 1. Prove that the finite-complement topology on $\mathbf{R}$ is coarser than the analytic topology.

Problem 2. Let $\mathcal{B}$ be the collection of intervals in $\mathbf{R}$ of the form $[a, b)$ (where $a < b$). It turns out that $\mathcal{B}$ is a basis. Show that the topology it generates is strictly finer than the analytic topology.

Problem 3. Give spaces X and Y and a continuous bijective map $f : X \to Y$ that is not a homeomorphism. You do not need to prove that f is continuous or bijective, but you should prove that it is not a homeomorphism.

Problem 4. Let $\mathcal{T}, \mathcal{T}'$ be topologies on X, with $\mathcal{T}$ coarser than $\mathcal{T}'$.

(1) If $f : X \to Y$ is continuous for $\mathcal{T}$, must f be continuous for $\mathcal{T}'$? If not, give a counterexample.

(2) If $g : Z \to X$ is continuous for $\mathcal{T}$, must g be continuous for $\mathcal{T}'$? If not, give a counterexample.

Problem 5. Prove that if $\mathcal{B}$ is a basis for a topological space X, and $A \subseteq X$, then

$$\{A \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on A.

Problem 6. Give an infinite subspace of $\mathbf{R}$ whose (subspace) topology is discrete. Justify your answer.

Problem 7. Give $\mathbf{R}^\omega$ the box topology and $X_n = \{x \in \mathbf{R}^\omega \mid x_i = 0 \text{ for } i > n\}$ the subspace topology. Show that $\mathbf{R}^n$, in its product topology, is homeomorphic to X_n.

Problem 8. Show that the formula

$$\eta(x, y) = \min\{1, |x - y|\}$$

defines a metric η on $\mathbf{R}$.

Problem 9. Show that $\mathbf{R}^\omega$ in its product topology is Hausdorff.

Problem 10. Show that any subspace of a Hausdorff space is also Hausdorff.

Problem 11. For any space X, subspace $Y \subseteq X$, and subset $A \subseteq Y$, show that

$$\mathrm{Int}_Y(A) \supseteq \mathrm{Int}_X(A).$$

Are these sets always equal? (Yes/No)

Problem 12. Let $p : \mathbf{R} \to \mathbf{R}_{\geq 0}$ be the map $p(x) = |x|$. It turns out that

$$\mathcal{B} = \{[0, c) \mid 0 < c\} \cup \{(a, b) \mid 0 < a < b\}$$

is a subset of the quotient topology on $\mathbf{R}_{\geq 0}$ induced by p. Show that $\mathcal{B}$ is actually a basis for this topology. *Hint:* Munkres Lemma 13.2.

Problem 13. View $x = (0, 0, 0, \ldots)$ and $y = (1, 2, 3, \ldots)$ as points of $\mathbf{R}^\omega$. Is there a path from x to y in the box topology on $\mathbf{R}^\omega$? *Hint:* Is $\mathbf{R}^\omega$ connected?

Problem 14. View

$$X = \mathbf{R} - \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\}$$

as a subspace of $\mathbf{R}$. Show that X is not locally connected. *Hint:* Show that the only connected open subset of X containing 0 is the set $\{0\}$, which isn't open.

Problem 15. Give examples of the following, with justification (possibly by citing theorems from lecture or Munkres):

(1) A connected subspace of $\mathbf{R}$ that is not compact.
(2) A compact subspace of $\mathbf{R}$ that is not connected.