MATH 199/249: INTRODUCTION TO GENERAL TOPOLOGY
PROBLEM SET #3

FALL 2026

Due Tuesday, November 24. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. We say that metrics $d, e\colon X^2 \to \mathbf{R}$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x,y) \le Ae(x,y) \text{ and } e(x,y) \le Bd(x,y) \text{ for all } x, y \in X.$$

Show that if d and e are equivalent, then they generate the same topology. (The converse is false.)

Hint: To show that the topology generated by d is coarser than that generated by e, it suffices to show that any open d-ball is a union of open e-balls. (Why?) And similarly with d and e interchanged.

Problem 2. Let $m\colon \mathbf{R}^2 \to \mathbf{R}$ be the multiplication map, $m(x,y) = xy$.

(1) Sketch the preimage under m of the open interval $(1,2)$. Is it analytically open in $\mathbf{R}^2$?
(2) Repeat (1) for the open interval $(-1,1)$.

Problem 3. Let a, b, c, d be real numbers with $a < b$ and $c < d$. Give an explicit homeomorphism between $[a,b)$ and $(c,d]$ in their subspace topologies inherited from analytic $\mathbf{R}$.

Problem 4. Endow $[-1,1]$ with the analytic topology: *i.e.*, the subspace topology it inherits from analytic $\mathbf{R}$. Determine which of the following sets are open in $[-1,1]$, and which are open in $\mathbf{R}$.

$$A = \{x \mid \tfrac{1}{2} < |x| < 1\},$$
$$B = \{x \mid \tfrac{1}{2} < |x| \le 1\},$$
$$C = \{x \mid \tfrac{1}{2} \le |x| < 1\},$$
$$D = \{x \mid \tfrac{1}{2} \le |x| \le 1\},$$
$$E = \{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbf{Z}_+\}.$$

Problem 5. For each set in Problem 4:

(1) If it is not open in $[-1,1]$, then write down its interior in $[-1,1]$.
(2) If it is not open in $\mathbf{R}$, then write down its interior in $\mathbf{R}$.

No need for rigorous justifications.

Problem 6. Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).

Problem 7. Show that X is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

Problem 8. Show that $\mathbf{Q}$ is not sequentially compact as a subspace of $\mathbf{R}$.

Problem 9. Give $\mathbf{R}$ the analytic topology. Let $K = \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\} \subseteq \mathbf{R}$.

(1) Show that $K \cup \{0\}$ is compact as a subspace of $\mathbf{R}$.
(2) Is K compact as a subspace of $\mathbf{R}$? Justify your answer.

Problem 10. Let X be Hausdorff, and let A, B be disjoint compact subspaces of X. Show that there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Problem 11. Show that if X is both discrete and compact, then X is finite.

Problem 12. Let X_n be the complement of the origin in $\mathbf{R}^n$. Show that X_n is not connected for $n = 1$, but connected for $n \geq 2$. *Hint:* For $n \geq 2$, construct explicit paths. Take care that your paths do not pass through the origin!

Problem 13. Show that no two of the spaces $(0, 1), (0, 1], [0, 1]$ are homeomorphic. *Hint:* What happens when a point is removed from each of these spaces?

Problem 14. Let $f \colon X \to X$ be continuous.

(1) Show that if $X = [0, 1]$, then f has a *fixed point*: that is, a point $x \in X$ such that $f(x) = x$.
(2) Show that the analogue of (1) fails for $X = [0, 1)$.

Problem 15. We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.
(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.

Problem 16. Give $\mathbf{R}$ the analytic topology and $\mathbf{Q}$ the subspace topology inherited from $\mathbf{R}$. Show that any continuous map from $\mathbf{R}$ into $\mathbf{Q}$ is constant.