MATH 199/249: INTRODUCTION TO GENERAL TOPOLOGY
PROBLEM SET #2

FALL 2026

Due Thursday, October 29. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. Let $f\colon X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Problem 2. Let X be a topological space, and let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is also open.

Problem 3. Let X be a set, and let $\mathcal{T}$ be a collection of subsets of X such that if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. Prove that

$$\text{for any } k > 0 \text{ and } U_1, \ldots, U_k \in \mathcal{T}, \quad \text{we have } U_1 \cap \cdots \cap U_k \in \mathcal{T}.$$

This problem shows that to check the axiom about finite intersections in the definition of a topology, it is enough to check pairwise intersections.

In the problems below, recall that a *basis* (for some topology) on a set X is a collection of subsets of X, say $\mathcal{B}$, with the following properties:

(1) X is the union of the elements of $\mathcal{B}$.
(2) For any $U, U' \in \mathcal{B}$ and $x \in U' \cap U''$, there exists $U'' \in \mathcal{B}$ such that $x \in U''$ and $U'' \subseteq U' \cap U'$.

In this case, the collection of sets

$$\{U \subseteq X \mid \text{for all } x \in U, \text{ there is some } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$$

forms a topology on X, called the topology *generated by $\mathcal{B}$*.

Problem 4. Show that the countable set

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$$

forms a basis for the analytic topology on $\mathbf{R}$. This shows that different bases (of different cardinalities) can generate the same topology.

Problem 5. Let $a\mathbf{Z} + b = \{aq + b \mid q \in \mathbf{Z}\}$, for any integers a and b. Let

$$\mathcal{B} = \{a\mathbf{Z} + b \mid a, b \in \mathbf{Z} \text{ with } a \neq 0\}.$$

Show that $\mathcal{B}$ forms a basis for some topology on $\mathbf{Z}$. We refer to the topology it generates as the *evenly-spaced topology*. *Hint:* If $x \in a\mathbf{Z} + b$, then $a\mathbf{Z} + b = a\mathbf{Z} + x$.

Problem 6. Endow $\mathbf{Z}$ with the evenly-spaced topology from Problem 5.

(1) Show that if there were finitely many (positive) primes, then $\{1, -1\}$ would be open. You may assume that any integer greater than 1 has a prime divisor.
(2) Deduce that there must be infinitely many primes.

Problem 7. Let X and Y be topological spaces. Show that:

(1) If the topology on X is discrete, then any map from X into Y is continuous.
(2) If the topology on Y is indiscrete, then any map from X into Y is continuous.
(3) Constant maps from X into Y are continuous, regardless of the topologies.

Problem 8. Show that the following topological spaces are homeomorphic:

(1) $\mathbf{R}$.
(2) $(0, \infty)$.
(3) $(0, 1)$.

Above, (1) is endowed with the analytic topology; (2) and (3) are endowed with the subspace topology. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 9. Let X, Y be topological spaces, and let $f \colon X \to Y$ be a continuous bijection.

(1) Give an example of X, Y, f where f is not a homeomorphism. *Hint:* Let X and Y have the same underlying set, but different topologies.
(2) Show that if $f(U)$ is open in Y for every open set U in X, then f must be a homeomorphism.

Recall that if X is a topological space and $A \subseteq X$ is any subset, then the *subspace topology* on A induced by X is the collection of sets

$$\{A \cap V \mid V \text{ open in } X\}.$$

When we equip A with this topology, we refer to the resulting topological space as the *subspace* of X formed by A.

Problem 10. Let X be a topological space, and let $A \subseteq X$ be a subset endowed with the subspace topology.

(1) Give an example where a subset of A is open in A, but not open in X.
(2) Now suppose that <u>A itself is open in X</u>. Prove that a subset of A is open in A if and only if it is open in X.

Problem 11. Endow $\mathbf{R}$ with the analytic topology, and

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology.

(1) Show that for all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.
(2) Show that $\{0\}$ is closed but not open.

Problem 12. Let $f \colon X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)} \colon f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y \colon Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y \colon Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Recall that if X_1 and X_2 are topological spaces, then the *product topology* on $X_1 \times X_2$ (inherited from X_1 and X_2) is the topology generated by the basis

$$\{U_1 \times U_2 \subseteq X_1 \times X_2 \mid U_i \text{ open in } X_i \text{ for } i = 1, 2\}.$$

Problem 13. Give an example showing that not every open set in $X_1 \times X_2$ need take the form $U_1 \times U_2$ for some open $U_1 \subseteq X_1$ and open $U_2 \subseteq X_2$. *Hint:* Take X_1, X_2 to be $\mathbf{R}$ in the analytic topology.

Problem 14. Show that the analytic topology on $\mathbf{R}^2$ coincides with the product topology that it inherits from $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

Problem 15. Show that if $f \colon A \to B$ and $g \colon C \to D$ are continuous maps, then $(f, g) \colon A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.

Problem 16. We define $\mathbf{R}^\omega$ to be the set of infinite sequences of real numbers indexed by $1, 2, 3, \ldots$ Thus:

$$\mathbf{R}^\omega = \{(x_i)_{i=1}^\infty \mid x_i \in \mathbf{R} \text{ for all } i\}.$$

Show that the collection of sets

$$\{U_1 \times U_2 \times \cdots \subseteq \mathbf{R}^\omega \mid U_i \text{ is open in } \mathbf{R} \text{ for all } i\}$$

forms a basis for some topology on $\mathbf{R}^\omega$. (Assume the Axiom of Choice.) It is called the *box topology*.