MATH 199/249: INTRODUCTION TO GENERAL TOPOLOGY
PROBLEM SET #1

FALL 2026

Due Thursday, September 24. Cite all sources, including people, books, papers, websites, and AI tools. Write up your answers in your own words.

Problem 1. Let z_1 and z_2 be complex numbers.

(1) Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.
(2) Prove that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Hint: Both parts can be solved using the triangle inequality.

Problem 2. Let $d, e \colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$d(\vec{u}, \vec{v}) = \max\{|v_1 - u_1|, |v_2 - u_2|\},$$
$$e(\vec{u}, \vec{v}) = \sqrt{|v_1 - u_1|^2 + 4|v_2 - u_2|^2}.$$

Decide which of these functions is a metric. Justify your answer. For each function that is a metric, sketch its unit circle at the origin.

Problem 3. For all positive integers n, let

$$a_n = \sqrt{n+1} - \sqrt{n}, \qquad b_n = \frac{1}{n}(\sqrt{n+1} - \sqrt{n}), \qquad c_n = n(\sqrt{n+1} - \sqrt{n}).$$

Decide which of these sequences converges, and if so, to which limit(s). Justify your answer.

Problem 4. Let $p_n = \frac{1}{n}$ and $q_n = \sum_{k=1}^{n} p_k$.

(1) Show that the sequence $\{p_n\}_{n=1}^{\infty}$ converges. What is its limit?
(2) Using induction on positive integers m, show that

$$q_{2^m} \geq 1 + \frac{1}{2}m.$$

(3) Using (2), deduce that the sequence $\{q_n\}_{n=1}^{\infty}$ diverges. Maybe you learned this in a calculus course?

Problem 5. Fix an integer $n > 0$.

(1) Show that for any $\vec{u}, \vec{v} \in \mathbf{R}^n$, we have

$$|\, \|\vec{v}\| - \|\vec{u}\| \,| \leq \|\vec{v} - \vec{u}\|.$$

(2) Using (1), show that the function $f \colon \mathbf{R}^n \to \mathbf{R}$ defined by $f(\vec{u}) = \|\vec{u}\|$ is analytically continuous.

Problem 6. Give an explicit function $f\colon \mathbf{R} \to \mathbf{R}$ with the given property. No need to justify your answer.

(1) f is analytically continuous, but not bounded.
(2) f is bounded, but not analytically continuous.
(3) f is analytically continuous and bounded and has no maximum.
(4) f is analytically continuous and bounded and $\lim_{x\to 0^+} f(1/x)$ does not exist.

Problem 7. Show that if $f, g\colon \mathbf{R}^n \to \mathbf{R}^m$ are both analytically continuous, then so is $f + g$.

Problem 8. Show that if $f\colon \mathbf{R}^n \to \mathbf{R}^m$ and $g\colon \mathbf{R}^m \to \mathbf{R}^\ell$ are analytically continuous, then so is $g \circ f$.

Problem 9. Show that any nonempty finite subset of $\mathbf{R}$ is analytically closed, but not analytically open.

Problem 10. Is the set of rational numbers analytically open in $\mathbf{R}$? Is it analytically closed?

Problem 11. Consider the following subset of $\mathbf{R}$:

$$K = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$

Show that $K \cup \{0\}$ is analytically closed. Is K analytically closed?

Problem 12. View $\mathbf{R}$ as the x-axis in $\mathbf{R}^2$. In this way, we may view subsets of $\mathbf{R}$ as subsets of $\mathbf{R}^2$.

(1) Show that no open subset of $\mathbf{R}$ remains analytically open in $\mathbf{R}^2$.
(2) Nonetheless, show that any closed subset of $\mathbf{R}$ remains analytically closed in $\mathbf{R}^2$. *Hint:* Show that the complement $\mathbf{R}^2 - \mathbf{R}$ is open.

Problem 13. Give an example of an analytically continuous, <u>non-constant</u> map $f\colon \mathbf{R} \to \mathbf{R}$ and an analytically open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* analytically open. *Hint:* Pick f to be polynomial.

Problem 14. Show that an infinite intersection of analytically open sets in $\mathbf{R}$ need not be analytically open. Then repeat with "union" and "closed" in place of "intersection" and "open".

Problem 15. Suppose that $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets, and the largest topology that is contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset.

Problem 16. Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

always forms a topology on X. What if we replace "countable" with "infinite" above?