MATH 199/249: INTRODUCTION TO GENERAL TOPOLOGY
EXAM 1 GUIDE

FALL 2026

Exam 1 will be held in-class on **Tuesday, September 15, 2026**. It will start shortly after 5:10 pm and end at 6:30 pm. It will be roughly five to six problems long.

You will be allowed to look at any notes on paper that you wrote prior to the exam. However, you will not be allowed to use electronic devices of any kind—including phones, computers, tablets, or audio devices—or any software, including AI tools.

What Could Appear

Metric Spaces.

- given a set X and function $d\colon X \times X \to \mathbf{R}$, how to check that d is a metric
- definition of the Euclidean metric on $\mathbf{R}^n$ for any integer $n > 0$
- examples where the triangle inequality for the Euclidean metric becomes an equality
- definition of the discrete metric on X for any set X
- statement of the Cauchy–Schwarz inequality and examples where it becomes an equality
- given a metric on $\mathbf{R}$ or $\mathbf{R}^2$, how to draw the set of points at a given distance from the origin

Sequences and Convergence.

- definition of convergence of a sequence $(p_n)_{n=1}^{\infty}$ to a point p ("for all $\epsilon > 0$, there is an integer $N \geq 1$ such that…")
- statement of the Archimedean principle for $\mathbf{R}$
- examples of convergent sequences in the Euclidean metric on $\mathbf{R}$, and their limits
- examples of divergent sequences in the Euclidean metric on $\mathbf{R}$
- examples where convergence of a sequence in a set X depends on the metric chosen on X

Sequential and Analytic Continuity.

- given metric spaces X, Y, a function $f\colon X \to Y$, and a point $p \in X$,
  - definition of sequential continuity of f at p
  - definition of Bolzano/analytic continuity of f at p ("for all $\epsilon > 0$, there is a $\delta > 0$ such that…")
- examples of continuous or discontinuous functions $f\colon \mathbf{R} \to \mathbf{R}$ in the Euclidean metrics on both domain and range

(Analytically) Open Sets.

- definition of (analytically) open balls in a metric space
- how to reformulate the definition of analytic continuity in terms of open balls
- definition of analytically open sets in a metric space
- how to prove that an open ball is, indeed, analytically open

Practice Problems

Try to hand-write a solution to each problem within 10–15 minutes. On the exam, you will not *need to use complete sentences, but the clearer your work, the more points you will earn.*

Problem 1. Show that the function $\eta\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by

$$\eta(x, y) = \min\{1, |x - y|\}$$

satisfies the triangle inequality.

Problem 2. Give examples of vectors $\vec{u}, \vec{v} \in \mathbf{R}^2$ in each of the following situations:

(1) $|\vec{u} \cdot \vec{v}| < \|\vec{u}\|\|\vec{v}\|$.
(2) $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|$.
(3) $\vec{u} \cdot \vec{v} = -\|\vec{u}\|\|\vec{v}\|$.

Problem 3. Using the Archimedean principle, show that the sequence $(a_n)_{n=1}^\infty$ defined by

$$a_n = \frac{(-1)^n}{2n - 1}$$

converges in $\mathbf{R}$.

Problem 4. Let $X = \{1, -1\}$. Show that if $(p_n)_{n=1}^\infty$ is given by $p_n = (-1)^n$, then $(p_n)_n$ diverges, regardless of the metric on X.

Problem 5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function that sends any real number x to the greatest integer less than or equal to x.

(1) Sketch the graph of $y = f(x)$, labeling the axes and a few points.
(2) Show that f is not sequentially continuous at $0 \in \mathbf{R}$, using the sequence $(p_n)_n$ given by $p_n = -1/n$.

Problem 6. Suppose that $g\colon X \to Y$ is analytically continuous at $p \in X$, and that Y has the discrete metric. Show that there is some $\delta > 0$ such that g is constant-valued on the open ball $B(p, \delta)$.

Problem 7. Recall the taxicab metric $d(\vec{u}, \vec{v}) = |v_1 - u_1| + |v_2 - u_2|$ on $\mathbf{R}^2$.

(1) Sketch the open ball $B((1, 1), 2) \subseteq \mathbf{R}^2$ in this metric.
(2) Check that $(1/2, 0) \in B((1, 1), 2)$.
(3) Give an explicit radius $\delta > 0$ such that $B((1/2, 0), \delta) \subseteq B((1, 1), 2)$.