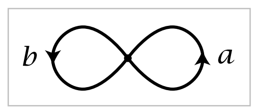
**20.** Suppose $f_t : X \to X$ is a homotopy such that $f_0$ and $f_1$ are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

# 1.2 Van Kampen's Theorem

The van Kampen theorem gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known. By systematic use of this theorem one can compute the fundamental groups of a very large number of spaces. We shall see for example that for every group $G$ there is a space $X_G$ whose fundamental group is isomorphic to $G$.

To give some idea of how one might hope to compute fundamental groups by decomposing spaces into simpler pieces, let us look at an example. Consider the space $X$ formed by two circles $A$ and $B$ intersecting in a single point, which we choose as the basepoint $x_0$. By our preceding calculations we know that $\pi_1(A)$ is infinite cyclic, generated by a loop $a$ that goes once around $A$. Similarly, $\pi_1(B)$ is a copy of $\mathbb{Z}$ generated by a loop $b$ going once around $B$. Each product of powers of $a$ and $b$ then gives an element of $\pi_1(X)$. For example, the product $a^5b^2a^{-3}ba^2$ is the loop that goes five times around $A$, then twice around $B$, then three times around $A$ in the opposite direction, then once around $B$, then twice around $A$. The set of all words like this consisting of powers of $a$ alternating with powers of $b$ forms a group usually denoted $\mathbb{Z} * \mathbb{Z}$. Multiplication in this group is defined just as one would expect, for example $(b^4a^5b^2a^{-3})(a^4b^{-1}ab^3) = b^4a^5b^2ab^{-1}ab^3$. The identity element is the empty word, and inverses are what they have to be, for example $(ab^2a^{-3}b^{-4})^{-1} = b^4a^3b^{-2}a^{-1}$. It would be very nice if such words in $a$ and $b$ corresponded exactly to elements of $\pi_1(X)$, so that $\pi_1(X)$ was isomorphic to the group $\mathbb{Z} * \mathbb{Z}$. The van Kampen theorem will imply that this is indeed the case.




Similarly, if $X$ is the union of three circles touching at a single point, the van Kampen theorem will imply that $\pi_1(X)$ is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, the group consisting of words in powers of three letters $a$, $b$, $c$. The generalization to a union of any number of circles touching at one point will also follow.

The group $\mathbb{Z} * \mathbb{Z}$ is an example of a general construction called the *free product* of groups. The statement of van Kampen's theorem will be in terms of free products, so before stating the theorem we will make an algebraic digression to describe the construction of free products in some detail.

## Free Products of Groups

Suppose one is given a collection of groups $G_\alpha$ and one wishes to construct a single group containing all these groups as subgroups. One way to do this would be to take the product group $\prod_\alpha G_\alpha$, whose elements can be regarded as the functions $\alpha \mapsto g_\alpha \in G_\alpha$. Or one could restrict to functions taking on nonidentity values at most finitely often, forming the direct sum $\bigoplus_\alpha G_\alpha$. Both these constructions produce groups containing all the $G_\alpha$'s as subgroups, but with the property that elements of different subgroups $G_\alpha$ commute with each other. In the realm of nonabelian groups this commutativity is unnatural, and so one would like a 'nonabelian' version of $\prod_\alpha G_\alpha$ or $\bigoplus_\alpha G_\alpha$. Since the sum $\bigoplus_\alpha G_\alpha$ is smaller and presumably simpler than $\prod_\alpha G_\alpha$, it should be easier to construct a nonabelian version of $\bigoplus_\alpha G_\alpha$, and this is what the free product $*_\alpha G_\alpha$ achieves.

Here is the precise definition. As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \ge 0$, where each letter $g_i$ belongs to a group $G_{\alpha_i}$ and is not the identity element of $G_{\alpha_i}$, and adjacent letters $g_i$ and $g_{i+1}$ belong to different groups $G_\alpha$, that is, $\alpha_i \ne \alpha_{i+1}$. Words satisfying these conditions are called *reduced*, the idea being that unreduced words can always be simplified to reduced words by writing adjacent letters that lie in the same $G_{\alpha_i}$ as a single letter and by canceling trivial letters. The empty word is allowed, and will be the identity element of $*_\alpha G_\alpha$. The group operation in $*_\alpha G_\alpha$ is juxtaposition, $(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$. This product may not be reduced, however: If $g_m$ and $h_1$ belong to the same $G_\alpha$, they should be combined into a single letter $(g_m h_1)$ according to the multiplication in $G_\alpha$, and if this new letter $g_m h_1$ happens to be the identity of $G_\alpha$, it should be canceled from the product. This may allow $g_{m-1}$ and $h_2$ to be combined, and possibly canceled too. Repetition of this process eventually produces a reduced word. For example, in the product $(g_1 \cdots g_m)(g_m^{-1} \cdots g_1^{-1})$ everything cancels and we get the identity element of $*_\alpha G_\alpha$, the empty word.

Verifying directly that this multiplication is associative would be rather tedious, but there is an indirect approach that avoids most of the work. Let $W$ be the set of reduced words $g_1 \cdots g_m$ as above, including the empty word. To each $g \in G_\alpha$ we associate the function $L_g : W \to W$ given by multiplication on the left, $L_g(g_1 \cdots g_m) = g g_1 \cdots g_m$ where we combine $g$ with $g_1$ if $g_1 \in G_\alpha$ to make $g g_1 \cdots g_m$ a reduced word. A key property of the association $g \mapsto L_g$ is the formula $L_{gg'} = L_g L_{g'}$ for $g, g' \in G_\alpha$, that is, $g(g'(g_1 \cdots g_m)) = (gg')(g_1 \cdots g_m)$. This special case of associativity follows rather trivially from associativity in $G_\alpha$. The formula $L_{gg'} = L_g L_{g'}$ implies that $L_g$ is invertible with inverse $L_{g^{-1}}$. Therefore the association $g \mapsto L_g$ defines a homomorphism from $G_\alpha$ to the group $P(W)$ of all permutations of $W$. More generally, we can define $L : W \to P(W)$ by $L(g_1 \cdots g_m) = L_{g_1} \cdots L_{g_m}$ for each reduced word $g_1 \cdots g_m$. This function $L$ is injective since the permutation $L(g_1 \cdots g_m)$ sends the empty word to $g_1 \cdots g_m$. The product operation in $W$ corresponds under $L$ to composition in $P(W)$, because of the relation $L_{gg'} = L_g L_{g'}$. Since composition in $P(W)$ is associative, we conclude that the product in $W$ is associative.

In particular, we have the free product $\mathbb{Z} * \mathbb{Z}$ as described earlier. This is an example of a *free group*, the free product of any number of copies of $\mathbb{Z}$, finite or infinite. The elements of a free group are uniquely representable as reduced words in powers of generators for the various copies of $\mathbb{Z}$, with one generator for each $\mathbb{Z}$, just as in the case of $\mathbb{Z} * \mathbb{Z}$. These generators are called a *basis* for the free group, and the number of basis elements is the *rank* of the free group. The abelianization of a free group is a free abelian group with basis the same set of generators, so since the rank of a free abelian group is well-defined, independent of the choice of basis, the same is true for the rank of a free group.

An interesting example of a free product that is not a free group is $\mathbb{Z}_2 * \mathbb{Z}_2$. This is like $\mathbb{Z} * \mathbb{Z}$ but simpler since $a^2 = e = b^2$, so powers of $a$ and $b$ are not needed, and $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of just the alternating words in $a$ and $b$: $a$, $b$, $ab$, $ba$, $aba$, $bab$, $abab$, $baba$, $ababa$, $\cdots$, together with the empty word. The structure of $\mathbb{Z}_2 * \mathbb{Z}_2$ can be elucidated by looking at the homomorphism $\varphi : \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2$ associating to each word its length mod 2. Obviously $\varphi$ is surjective, and its kernel consists of the words of even length. These form an infinite cyclic subgroup generated by $ab$ since $ba = (ab)^{-1}$ in $\mathbb{Z}_2 * \mathbb{Z}_2$. In fact, $\mathbb{Z}_2 * \mathbb{Z}_2$ is the semi-direct product of the subgroups $\mathbb{Z}$ and $\mathbb{Z}_2$ generated by $ab$ and $a$, with the conjugation relation $a(ab)a^{-1} = (ab)^{-1}$. This group is sometimes called the infinite dihedral group.

For a general free product $*_\alpha G_\alpha$, each group $G_\alpha$ is naturally identified with a subgroup of $*_\alpha G_\alpha$, the subgroup consisting of the empty word and the nonidentity one-letter words $g \in G_\alpha$. From this viewpoint the empty word is the common identity element of all the subgroups $G_\alpha$, which are otherwise disjoint. A consequence of associativity is that any product $g_1 \cdots g_m$ of elements $g_i$ in the groups $G_\alpha$ has a unique reduced form, the element of $*_\alpha G_\alpha$ obtained by performing the multiplications in any order. Any sequence of reduction operations on an unreduced product $g_1 \cdots g_m$, combining adjacent letters $g_i$ and $g_{i+1}$ that lie in the same $G_\alpha$ or canceling a $g_i$ that is the identity, can be viewed as a way of inserting parentheses into $g_1 \cdots g_m$ and performing the resulting sequence of multiplications. Thus associativity implies that any two sequences of reduction operations performed on the same unreduced word always yield the same reduced word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$ extends uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$. Namely, the value of $\varphi$ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define $\varphi$ gives a well-defined homomorphism since the process of reducing an unreduced product in $*_\alpha G_\alpha$ does not affect its image under $\varphi$. For example, for a free product $G * H$ the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

## The van Kampen Theorem

Suppose a space $X$ is decomposed as the union of a collection of path-connected open subsets $A_\alpha$, each of which contains the basepoint $x_0 \in X$. By the remarks in the preceding paragraph, the homomorphisms $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that $\Phi$ is very often surjective, but we can expect $\Phi$ to have a nontrivial kernel in general. For if $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both these compositions being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of $\Phi$ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$. Van Kampen's theorem asserts that under fairly broad hypotheses this gives a full description of $\Phi$:

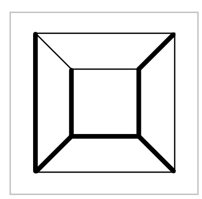
**Theorem 1.20.** *If $X$ is the union of path-connected open sets $A_\alpha$ each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of $\Phi$ is the normal subgroup $N$ generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence $\Phi$ induces an isomorphism $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$.*

**Example 1.21: Wedge Sums.** In Chapter 0 we defined the wedge sum $\bigvee_\alpha X_\alpha$ of a collection of spaces $X_\alpha$ with basepoints $x_\alpha \in X_\alpha$ to be the quotient space of the disjoint union $\coprod_\alpha X_\alpha$ in which all the basepoints $x_\alpha$ are identified to a single point. If each $x_\alpha$ is a deformation retract of an open neighborhood $U_\alpha$ in $X_\alpha$, then $X_\alpha$ is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct $A_\alpha$'s is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Van Kampen's theorem then implies that $\Phi : *_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ is an isomorphism.

Thus for a wedge sum $\bigvee_\alpha S^1_\alpha$ of circles, $\pi_1(\bigvee_\alpha S^1_\alpha)$ is a free group, the free product of copies of $\mathbb{Z}$, one for each circle $S^1_\alpha$. In particular, $\pi_1(S^1 \vee S^1)$ is the free group $\mathbb{Z} * \mathbb{Z}$, as in the example at the beginning of this section.

It is true more generally that the fundamental group of any connected graph is free, as we show in §1.A. Here is an example illustrating the general technique.

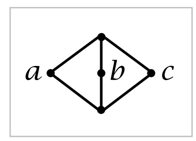
**Example 1.22.** Let $X$ be the graph shown in the figure, consisting of the twelve edges of a cube. The seven heavily shaded edges form a maximal tree $T \subset X$, a contractible subgraph containing all the vertices of $X$. We claim that $\pi_1(X)$ is the free product of five copies of $\mathbb{Z}$, one for each edge not in $T$. To deduce this from van Kampen's theorem, choose for each edge $e_\alpha$ of $X - T$ an open neighborhood $A_\alpha$ of $T \cup e_\alpha$ in $X$ that deformation retracts onto $T \cup e_\alpha$. The intersection of two or more $A_\alpha$'s deformation retracts onto $T$, hence is contractible. The $A_\alpha$'s form a cover of $X$ satisfying the hypotheses of van Kampen's theorem, and since the intersection of



any two of them is simply-connected we obtain an isomorphism $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)$. Each $A_\alpha$ deformation retracts onto a circle, so $\pi_1(X)$ is free on five generators, as claimed. As explicit generators we can choose for each edge $e_\alpha$ of $X - T$ a loop $f_\alpha$ that starts at a basepoint in $T$, travels in $T$ to one end of $e_\alpha$, then across $e_\alpha$, then back to the basepoint along a path in $T$.

Van Kampen's theorem is often applied when there are just two sets $A_\alpha$ and $A_\beta$ in the cover of $X$, so the condition on triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ is superfluous and one obtains an isomorphism $\pi_1(X) \approx (\pi_1(A_\alpha) * \pi_1(A_\beta))/N$, under the assumption that $A_\alpha \cap A_\beta$ is path-connected. The proof in this special case is virtually identical with the proof in the general case, however.

One can see that the intersections $A_\alpha \cap A_\beta$ need to be path-connected by considering the example of $S^1$ decomposed as the union of two open arcs. In this case $\Phi$ is not surjective. For an example showing that triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ need to be path-connected, let $X$ be the suspension of three points $a$, $b$, $c$, and let $A_\alpha, A_\beta$, and $A_\gamma$ be the complements of these three points. The theorem does apply to the covering $\{A_\alpha, A_\beta\}$, so there are isomorphisms $\pi_1(X) \approx \pi_1(A_\alpha) * \pi_1(A_\beta) \approx \mathbb{Z} * \mathbb{Z}$ since $A_\alpha \cap A_\beta$ is contractible. If we tried to use the covering $\{A_\alpha, A_\beta, A_\gamma\}$, which has each of the twofold intersections path-connected but not the triple intersection, then we would get $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, but this is not isomorphic to $\mathbb{Z} * \mathbb{Z}$ since it has a different abelianization.

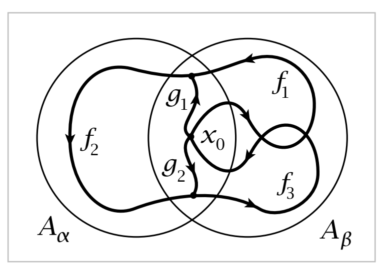



**Proof of van Kampen's theorem**: First we consider surjectivity of $\Phi$. Given a loop $f: I \to X$ at the basepoint $x_0$, we claim there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $I$ such that each subinterval $[s_{i-1}, s_i]$ is mapped by $f$ to a single $A_\alpha$. Namely, since $f$ is continuous, each $s \in I$ has an open neighborhood $V_s$ in $I$ mapped by $f$ to some $A_\alpha$. We may in fact take $V_s$ to be an interval whose closure is mapped to a single $A_\alpha$. Compactness of $I$ implies that a finite number of these intervals cover $I$. The endpoints of this finite set of intervals then define the desired partition of $I$.

Denote the $A_\alpha$ containing $f([s_{i-1}, s_i])$ by $A_i$, and let $f_i$ be the path obtained by restricting $f$ to $[s_{i-1}, s_i]$. Then $f$ is the composition $f_1 \cdot \cdots \cdot f_m$ with $f_i$ a path in $A_i$. Since we assume $A_i \cap A_{i+1}$ is path-connected, we may choose a path $g_i$ in $A_i \cap A_{i+1}$ from $x_0$ to the point $f(s_i) \in A_i \cap A_{i+1}$. Consider the loop

$$(f_1 \cdot \overline{g}_1) \cdot (g_1 \cdot f_2 \cdot \overline{g}_2) \cdot (g_2 \cdot f_3 \cdot \overline{g}_3) \cdot \cdots \cdot (g_{m-1} \cdot f_m)$$

which is homotopic to $f$. This loop is a composition of loops each lying in a single $A_i$, the loops indicated by the parentheses. Hence $[f]$ is in the image of $\Phi$, and $\Phi$ is surjective.




The harder part of the proof is to show that the kernel of $\Phi$ is $N$. It may clarify matters to introduce some terminology. By a *factorization* of an element $[f] \in \pi_1(X)$ we shall mean a formal product $[f_1] \cdots [f_k]$ where:

- Each $f_i$ is a loop in some $A_\alpha$ at the basepoint $x_0$, and $[f_i] \in \pi_1(A_\alpha)$ is the homotopy class of $f_i$.
- The loop $f$ is homotopic to $f_1 \cdot \cdots \cdot f_k$ in $X$.

A factorization of $[f]$ is thus a word in $*_\alpha \pi_1(A_\alpha)$, possibly unreduced, that is mapped to $[f]$ by $\Phi$. The proof of surjectivity of $\Phi$ showed that every $[f] \in \pi_1(X)$ has a factorization.

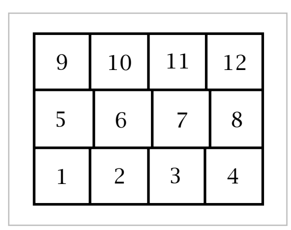
We will be concerned now with the uniqueness of factorizations. Call two factorizations of $[f]$ *equivalent* if they are related by a sequence of the following two sorts of moves or their inverses:

- Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i \cdot f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(A_\alpha)$.
- Regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in the group $\pi_1(A_\beta)$ rather than $\pi_1(A_\alpha)$ if $f_i$ is a loop in $A_\alpha \cap A_\beta$.

The first move does not change the element of $*_\alpha \pi_1(A_\alpha)$ defined by the factorization. The second move does not change the image of this element in the quotient group $Q = *_\alpha \pi_1(A_\alpha)/N$, by the definition of $N$. So equivalent factorizations give the same element of $Q$.

If we can show that any two factorizations of $[f]$ are equivalent, this will say that the map $Q \to \pi_1(X)$ induced by $\Phi$ is injective, hence the kernel of $\Phi$ is exactly $N$, and the proof will be complete.

Let $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_\ell']$ be two factorizations of $[f]$. The composed paths $f_1 \cdot \cdots \cdot f_k$ and $f_1' \cdot \cdots \cdot f_\ell'$ are then homotopic, so let $F : I \times I \to X$ be a homotopy from $f_1 \cdot \cdots \cdot f_k$ to $f_1' \cdot \cdots \cdot f_\ell'$. There exist partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by $F$ into a single $A_\alpha$, which we label $A_{ij}$. These partitions may be obtained by covering $I \times I$ by finitely many rectangles $[a,b] \times [c,d]$ each mapping to a single $A_\alpha$, using a compactness argument, then partitioning $I \times I$ by the union of all the horizontal and vertical lines containing edges of these rectangles. We may assume the $s$-partition subdivides the partitions giving the products $f_1 \cdot \cdots \cdot f_k$ and $f_1' \cdot \cdots \cdot f_\ell'$. Since $F$ maps a neighborhood of $R_{ij}$ to $A_{ij}$, we may perturb the vertical sides of the rectangles $R_{ij}$ so that each point of $I \times I$ lies in at most three $R_{ij}$'s. We may assume there are at least three rows of rectangles, so we can do this perturbation just on the rectangles in the intermediate rows, leaving the top and bottom rows unchanged. Let us relabel the new rectangles $R_1, R_2, \cdots, R_{mn}$, ordering them as in the figure.

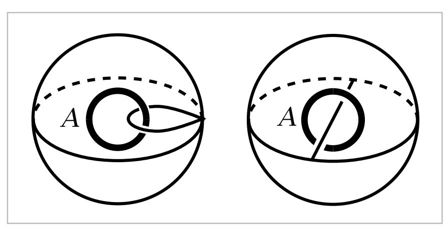


If $\gamma$ is a path in $I\times I$ from the left edge to the right edge, then the restriction $F\,|\,\gamma$ is a loop at the basepoint $x_0$ since $F$ maps both the left and right edges of $I\times I$ to $x_0$. Let $\gamma_r$ be the path separating the first $r$ rectangles $R_1,\cdots,R_r$ from the remaining rectangles. Thus $\gamma_0$ is the bottom edge of $I\times I$ and $\gamma_{mn}$ is the top edge. We pass from $\gamma_r$ to $\gamma_{r+1}$ by pushing across the rectangle $R_{r+1}$.

Let us call the corners of the $R_r$'s *vertices*. For each vertex $v$ with $F(v)\neq x_0$, let $g_v$ be a path from $x_0$ to $F(v)$. We can choose $g_v$ to lie in the intersection of the two or three $A_{ij}$'s corresponding to the $R_r$'s containing $v$ since we assume the intersection of any two or three $A_{ij}$'s is path-connected. If we insert into $F\,|\,\gamma_r$ the appropriate paths $\overline{g}_v g_v$ at successive vertices, as in the proof of surjectivity of $\Phi$, then we obtain a factorization of $[F\,|\,\gamma_r]$ by regarding the loop corresponding to a horizontal or vertical segment between adjacent vertices as lying in the $A_{ij}$ for either of the $R_s$'s containing the segment. Different choices of these containing $R_s$'s change the factorization of $[F\,|\,\gamma_r]$ to an equivalent factorization. Furthermore, the factorizations associated to successive paths $\gamma_r$ and $\gamma_{r+1}$ are equivalent since pushing $\gamma_r$ across $R_{r+1}$ to $\gamma_{r+1}$ changes $F\,|\,\gamma_r$ to $F\,|\,\gamma_{r+1}$ by a homotopy within the $A_{ij}$ corresponding to $R_{r+1}$, and we can choose this $A_{ij}$ for all the segments of $\gamma_r$ and $\gamma_{r+1}$ in $R_{r+1}$.

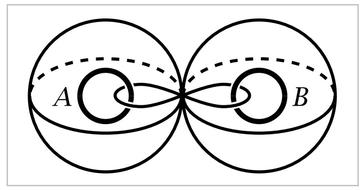
We can arrange that the factorization associated to $\gamma_0$ is equivalent to the factorization $[f_1]\cdots[f_k]$ by choosing the path $g_v$ for each vertex $v$ along the lower edge of $I\times I$ to lie not just in the two $A_{ij}$'s corresponding to the $R_s$'s containing $v$, but also to lie in the $A_\alpha$ for the $f_i$ containing $v$ in its domain. In case $v$ is the common endpoint of the domains of two consecutive $f_i$'s we have $F(v)=x_0$, so there is no need to choose a $g_v$. In similar fashion we may assume that the factorization associated to the final $\gamma_{mn}$ is equivalent to $[f_1']\cdots[f_\ell']$. Since the factorizations associated to all the $\gamma_r$'s are equivalent, we conclude that the factorizations $[f_1]\cdots[f_k]$ and $[f_1']\cdots[f_\ell']$ are equivalent. □

**Example 1.23: Linking of Circles.** We can apply van Kampen's theorem to calculate the fundamental groups of three spaces discussed in the introduction to this chapter, the complements in $\mathbb{R}^3$ of a single circle, two unlinked circles, and two linked circles.

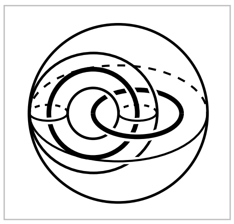



The complement $\mathbb{R}^3-A$ of a single circle $A$ deformation retracts onto a wedge sum $S^1\vee S^2$ embedded in $\mathbb{R}^3-A$ as shown in the first of the two figures at the right. It may be easier to see that $\mathbb{R}^3-A$ deformation retracts onto the union of $S^2$ with a diameter, as in the second figure, where points outside $S^2$ deformation retract onto $S^2$, and points inside $S^2$ and not in $A$ can be pushed away from $A$ toward $S^2$ or the diameter. Having this deformation retraction in mind, one can then see how it must be modified if the two endpoints of the diameter are gradually moved toward each other along the equator until they coincide, forming the $S^1$ summand of $S^1\vee S^2$. Another way of seeing the deformation retraction of $\mathbb{R}^3 - A$ onto $S^1 \vee S^2$ is to note first that an open $\varepsilon$-neighborhood of $S^1 \vee S^2$ obviously deformation retracts onto $S^1 \vee S^2$ if $\varepsilon$ is sufficiently small. Then observe that this neighborhood is homeomorphic to $\mathbb{R}^3 - A$ by a homeomorphism that is the identity on $S^1 \vee S^2$. In fact, the neighborhood can be gradually enlarged by homeomorphisms until it becomes all of $\mathbb{R}^3 - A$.

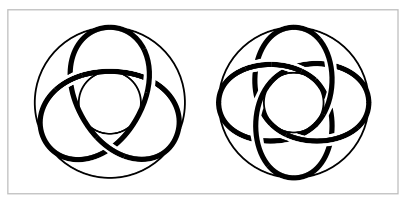
In any event, once we see that $\mathbb{R}^3 - A$ deformation retracts to $S^1 \vee S^2$, then we immediately obtain isomorphisms $\pi_1(\mathbb{R}^3 - A) \approx \pi_1(S^1 \vee S^2) \approx \mathbb{Z}$ since $\pi_1(S^2) = 0$.




In similar fashion, the complement $\mathbb{R}^3 - (A \cup B)$ of two unlinked circles $A$ and $B$ deformation retracts onto $S^1 \vee S^1 \vee S^2 \vee S^2$, as in the figure to the right. From this we get $\pi_1(\mathbb{R}^3 - (A \cup B)) \approx \mathbb{Z} * \mathbb{Z}$. On the other hand, if $A$ and $B$ are linked, then $\mathbb{R}^3 - (A \cup B)$ deformation retracts onto the wedge sum of $S^2$ and a torus $S^1 \times S^1$ separating $A$ and $B$, as shown in the figure to the left, hence $\pi_1(\mathbb{R}^3 - (A \cup B)) \approx \pi_1(S^1 \times S^1) \approx \mathbb{Z} \times \mathbb{Z}$.



**Example 1.24: Torus Knots**. For relatively prime positive integers $m$ and $n$, the **torus knot** $K = K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$, $f(z) = (z^m, z^n)$, where the torus $S^1 \times S^1$ is embedded in $\mathbb{R}^3$ in the standard way. The knot $K$ winds around the torus a total of $m$ times in the longitudinal direction and $n$ times in the meridional direction, as shown in the figure for the cases $(m,n) = (2,3)$ and $(3,4)$. One needs to assume that $m$ and $n$ are relatively prime in order for the map $f$ to be injective. Without this assumption $f$ would be $d$–to–1 where $d$ is the greatest common divisor of $m$ and $n$, and the image of $f$ would be the knot $K_{m/d,n/d}$. One could also allow negative values for $m$ or $n$, but this would only change $K$ to a mirror-image knot.

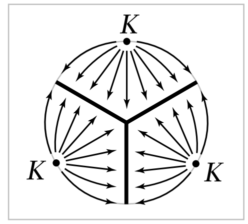


Let us compute $\pi_1(\mathbb{R}^3 - K)$. It is slightly easier to do the calculation with $\mathbb{R}^3$ replaced by its one-point compactification $S^3$. An application of van Kampen's theorem shows that this does not affect $\pi_1$. Namely, write $S^3 - K$ as the union of $\mathbb{R}^3 - K$ and an open ball $B$ formed by the compactification point together with the complement of a large closed ball in $\mathbb{R}^3$ containing $K$. Both $B$ and $B \cap (\mathbb{R}^3 - K)$ are simply-connected, the latter space being homeomorphic to $S^2 \times \mathbb{R}$. Hence van Kampen's theorem implies that the inclusion $\mathbb{R}^3 - K \hookrightarrow S^3 - K$ induces an isomorphism on $\pi_1$.

We compute $\pi_1(S^3 - K)$ by showing that it deformation retracts onto a 2-dimensional complex $X = X_{m,n}$ homeomorphic to the quotient space of a cylinder $S^1 \times I$ under the identifications $(z,0) \sim (e^{2\pi i/m}z, 0)$ and $(z,1) \sim (e^{2\pi i/n}z, 1)$. If we let $X_m$ and $X_n$ be the two halves of $X$ formed by the quotients of $S^1 \times [0, {}^1/_2]$ and $S^1 \times [{}^1/_2, 1]$, then $X_m$ and $X_n$ are the mapping cylinders of $z \mapsto z^m$ and $z \mapsto z^n$. The intersection

$X_m \cap X_n$ is the circle $S^1 \times \{1/2\}$, the domain end of each mapping cylinder.

To obtain an embedding of $X$ in $S^3 - K$ as a deformation retract we will use the standard decomposition of $S^3$ into two solid tori $S^1 \times D^2$ and $D^2 \times S^1$, the result of regarding $S^3$ as $\partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2$. Geometrically, the first solid torus $S^1 \times D^2$ can be identified with the compact region in $\mathbb{R}^3$ bounded by the standard torus $S^1 \times S^1$ containing $K$, and the second solid torus $D^2 \times S^1$ is then the closure of the complement of the first solid torus, together with the compactification point at infinity. Notice that meridional circles in $S^1 \times S^1$ bound disks in the first solid torus, while it is longitudinal circles that bound disks in the second solid torus.




In the first solid torus, $K$ intersects each of the meridian circles $\{x\} \times \partial D^2$ in $m$ equally spaced points, as indicated in the figure at the right, which shows a meridian disk $\{x\} \times D^2$. These $m$ points can be separated by a union of $m$ radial line segments. Letting $x$ vary, these radial segments then trace out a copy of the mapping cylinder $X_m$ in the first solid torus. Symmetrically, there is a copy of the other mapping cylinder $X_n$ in the second solid torus. The complement of $K$ in the first solid torus deformation retracts onto $X_m$ by flowing within each meridian disk as shown. In similar fashion the complement of $K$ in the second solid torus deformation retracts onto $X_n$. These two deformation retractions do not agree on their common domain of definition $S^1 \times S^1 - K$, but this is easy to correct by distorting the flows in the two solid tori so that in $S^1 \times S^1 - K$ both flows are orthogonal to $K$. After this modification we now have a well-defined deformation retraction of $S^3 - K$ onto $X$. Another way of describing the situation would be to say that for an open $\varepsilon$-neighborhood $N$ of $K$ bounded by a torus $T$, the complement $S^3 - N$ is the mapping cylinder of a map $T \to X$.

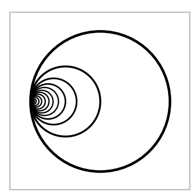
To compute $\pi_1(X)$ we apply van Kampen's theorem to the decomposition of $X$ as the union of $X_m$ and $X_n$, or more properly, open neighborhoods of these two sets that deformation retract onto them. Both $X_m$ and $X_n$ are mapping cylinders that deformation retract onto circles, and $X_m \cap X_n$ is a circle, so all three of these spaces have fundamental group $\mathbb{Z}$. A loop in $X_m \cap X_n$ representing a generator of $\pi_1(X_m \cap X_n)$ is homotopic in $X_m$ to a loop representing $m$ times a generator, and in $X_n$ to a loop representing $n$ times a generator. Van Kampen's theorem then says that $\pi_1(X)$ is the quotient of the free group on generators $a$ and $b$ obtained by factoring out the normal subgroup generated by the element $a^m b^{-n}$.

Let us denote by $G_{m,n}$ this group $\pi_1(X_{m,n})$ defined by two generators $a$ and $b$ and one relation $a^m = b^n$. If $m$ or $n$ is $1$, then $G_{m,n}$ is infinite cyclic since in these cases the relation just expresses one generator as a power of the other. To describe the structure of $G_{m,n}$ when $m, n > 1$ let us first compute the center of $G_{m,n}$, the subgroup consisting of elements that commute with all elements of $G_{m,n}$. The element $a^m = b^n$ commutes with $a$ and $b$, so the cyclic subgroup $C$ generated by this element lies in the center. In particular, $C$ is a normal subgroup, so we can pass to the quotient group $G_{m,n}/C$, which is the free product $\mathbb{Z}_m * \mathbb{Z}_n$. According to Exercise 1 at the end of this section, a free product of nontrivial groups has trivial center. From this it follows that $C$ is exactly the center of $G_{m,n}$. As we will see in Example 1.44, the elements $a$ and $b$ have infinite order in $G_{m,n}$, so $C$ is infinite cyclic, but we will not need this fact here.

We will show now that the integers $m$ and $n$ are uniquely determined by the group $\mathbb{Z}_m * \mathbb{Z}_n$, hence also by $G_{m,n}$. The abelianization of $\mathbb{Z}_m * \mathbb{Z}_n$ is $\mathbb{Z}_m \times \mathbb{Z}_n$, of order $mn$, so the product $mn$ is uniquely determined by $\mathbb{Z}_m * \mathbb{Z}_n$. To determine $m$ and $n$ individually, we use another assertion from Exercise 1 at the end of the section, that all torsion elements of $\mathbb{Z}_m * \mathbb{Z}_n$ are conjugate to elements of the subgroups $\mathbb{Z}_m$ and $\mathbb{Z}_n$, hence have order dividing $m$ or $n$. Thus the maximum order of torsion elements of $\mathbb{Z}_m * \mathbb{Z}_n$ is the larger of $m$ and $n$. The larger of these two numbers is therefore uniquely determined by the group $\mathbb{Z}_m * \mathbb{Z}_n$, hence also the smaller since the product is uniquely determined.

The preceding analysis of $\pi_1(X_{m,n})$ did not need the assumption that $m$ and $n$ are relatively prime, which was used only to relate $X_{m,n}$ to torus knots. An interesting fact is that $X_{m,n}$ can be embedded in $\mathbb{R}^3$ only when $m$ and $n$ are relatively prime. This is shown in the remarks following Corollary 3.45. For example, $X_{2,2}$ is the Klein bottle since it is the union of two copies of the Möbius band $X_2$ with their boundary circles identified, so this nonembeddability statement generalizes the fact that the Klein bottle cannot be embedded in $\mathbb{R}^3$.

An algorithm for computing a presentation for $\pi_1(\mathbb{R}^3 - K)$ for an arbitrary smooth or piecewise linear knot $K$ is described in the exercises, but the problem of determining when two of these fundamental groups are isomorphic is generally much more difficult than in the special case of torus knots.



**Example 1.25: The Shrinking Wedge of Circles.** Consider the subspace $X \subset \mathbb{R}^2$ that is the union of the circles $C_n$ of radius $1/n$ and center $(1/n, 0)$ for $n = 1, 2, \cdots$. At first glance one might confuse $X$ with the wedge sum of an infinite sequence of circles, but we will show that $X$ has a much larger fundamental group than the wedge sum. Consider the retractions $r_n : X \to C_n$ collapsing all $C_i$'s except $C_n$ to the origin. Each $r_n$ induces a surjection $\rho_n : \pi_1(X) \to \pi_1(C_n) \approx \mathbb{Z}$, where we take the origin as the basepoint. The product of the $\rho_n$'s is a homomorphism $\rho : \pi_1(X) \to \prod_\infty \mathbb{Z}$ to the direct product (not the direct sum) of infinitely many copies of $\mathbb{Z}$, and $\rho$ is surjective since for every sequence of integers $k_n$ we can construct a loop $f : I \to X$ that wraps $k_n$ times around $C_n$ in the time interval $[1 - 1/n, 1 - 1/{n+1}]$. This infinite composition of loops is certainly continuous at each time less than $1$, and it is continuous at time $1$ since every neighborhood of the basepoint in $X$ contains all but finitely many of the circles $C_n$. Since $\pi_1(X)$ maps onto the uncountable group $\prod_\infty \mathbb{Z}$, it is uncountable.

On the other hand, the fundamental group of a wedge sum of countably many circles is countably generated, hence countable.

The group $\pi_1(X)$ is actually far more complicated than $\prod_\infty \mathbb{Z}$. For one thing, it is nonabelian, since the retraction $X \to C_1 \cup \cdots \cup C_n$ that collapses all the circles smaller than $C_n$ to the basepoint induces a surjection from $\pi_1(X)$ to a free group on $n$ generators. For a complete description of $\pi_1(X)$ see [Cannon & Conner 2000].

It is a theorem of [Shelah 1988] that for a path-connected, locally path-connected compact metric space $X$, $\pi_1(X)$ is either finitely generated or uncountable.

## Applications to Cell Complexes

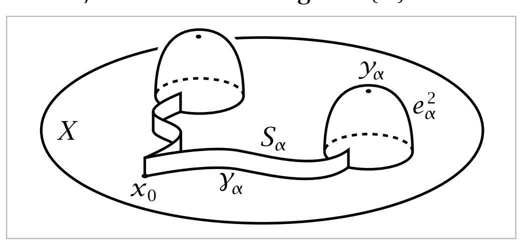
For the remainder of this section we shall be interested in 2-dimensional cell complexes, analyzing how the fundamental group is affected by attaching 2-cells. According to an exercise at the end of this section, attaching cells of higher dimension has no effect on $\pi_1$, so all the interest lies in how the 2-cells are attached.

Suppose we attach a collection of 2-cells $e^2_\alpha$ to a path-connected space $X$ via maps $\varphi_\alpha : S^1 \to X$, producing a space $Y$. If $s_0$ is a basepoint of $S^1$ then $\varphi_\alpha$ determines a loop at $\varphi_\alpha(s_0)$ that we shall call $\varphi_\alpha$, even though technically loops are maps $I \to X$ rather than $S^1 \to X$. For different $\alpha$'s the basepoints $\varphi_\alpha(s_0)$ of these loops $\varphi_\alpha$ may not all coincide. To remedy this, choose a basepoint $x_0 \in X$ and a path $\gamma_\alpha$ in $X$ from $x_0$ to $\varphi_\alpha(s_0)$ for each $\alpha$. Then $\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha$ is a loop at $x_0$. This loop may not be nullhomotopic in $X$, but it will certainly be nullhomotopic after the cell $e^2_\alpha$ is attached. Thus the normal subgroup $N \subset \pi_1(X, x_0)$ generated by all the loops $\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha$ for varying $\alpha$ lies in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $X \hookrightarrow Y$.

**Proposition 1.26.** *The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is $N$. Thus $\pi_1(Y) \approx \pi_1(X)/N$.*

It follows that $N$ is independent of the choice of the paths $\gamma_\alpha$, but this can also be seen directly: If we replace $\gamma_\alpha$ by another path $\eta_\alpha$ having the same endpoints, then $\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha$ changes to $\eta_\alpha \varphi_\alpha \overline{\eta}_\alpha = (\eta_\alpha \overline{\gamma}_\alpha)\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha(\gamma_\alpha \overline{\eta}_\alpha)$, so $\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha$ and $\eta_\alpha \varphi_\alpha \overline{\eta}_\alpha$ define conjugate elements of $\pi_1(X, x_0)$.

**Proof**: Let us expand $Y$ to a slightly larger space $Z$ that deformation retracts onto $Y$ and is more convenient for applying van Kampen's theorem. The space $Z$ is obtained from $Y$ by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge $I \times \{0\}$ attached along $\gamma_\alpha$, the right edge $\{1\} \times I$ attached along an arc in $e^2_\alpha$, and all the left edges $\{0\} \times I$ of the different strips identified together. The top edges of the strips are not attached to anything, and this allows us to deformation retract $Z$ onto $Y$.



In each cell $e^2_\alpha$ choose a point $y_\alpha$ not in the arc along which $S_\alpha$ is attached. Let $A = Z - \bigcup_\alpha \{y_\alpha\}$ and let $B = Z - X$. Then $A$ deformation retracts onto $X$, and $B$ is contractible. Since $\pi_1(B) = 0$, van Kampen's theorem applied to the cover $\{A, B\}$ says that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(A)$ by the normal subgroup generated by the image of the map $\pi_1(A\cap B) \to \pi_1(A)$. So it remains only to see that $\pi_1(A\cap B)$ is generated by the loops $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$, or rather by loops in $A\cap B$ homotopic to these loops. This can be shown by another application of van Kampen's theorem, this time to the cover of $A\cap B$ by the open sets $A_\alpha = A\cap B - \bigcup_{\beta\neq\alpha} e^2_\beta$. Since $A_\alpha$ deformation retracts onto a circle in $e^2_\alpha - \{y_\alpha\}$, we have $\pi_1(A_\alpha) \approx \mathbb{Z}$ generated by a loop homotopic to $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$, and the result follows. □

As a first application we compute the fundamental group of the orientable surface $M_g$ of genus $g$. This has a cell structure with one 0-cell, $2g$ 1-cells, and one 2-cell, as we saw in Chapter 0. The 1-skeleton is a wedge sum of $2g$ circles, with fundamental group free on $2g$ generators. The 2-cell is attached along the loop given by the product of the commutators of these generators, say $[a_1, b_1]\cdots[a_g, b_g]$. Therefore

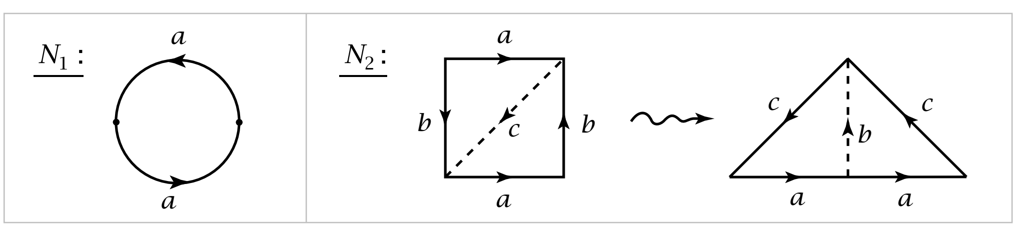
$$\pi_1(M_g) \approx \langle\, a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1]\cdots[a_g, b_g] \,\rangle$$

where $\langle\, g_\alpha \mid r_\beta \,\rangle$ denotes the group with generators $g_\alpha$ and relators $r_\beta$, in other words, the free group on the generators $g_\alpha$ modulo the normal subgroup generated by the words $r_\beta$ in these generators.

**Corollary 1.27.** *The surface $M_g$ is not homeomorphic, or even homotopy equivalent, to $M_h$ if $g \neq h$.*

**Proof**: The abelianization of $\pi_1(M_g)$ is the direct sum of $2g$ copies of $\mathbb{Z}$. So if $M_g \simeq M_h$ then $\pi_1(M_g) \approx \pi_1(M_h)$, hence the abelianizations of these groups are isomorphic, which implies $g = h$. □

Nonorientable surfaces can be treated in the same way. If we attach a 2-cell to the wedge sum of $g$ circles by the word $a_1^2\cdots a_g^2$ we obtain a nonorientable surface $N_g$. For example, $N_1$ is the projective plane $\mathbb{R}P^2$, the quotient of $D^2$ with antipodal points of $\partial D^2$ identified. And $N_2$ is the Klein bottle, though the more usual representation



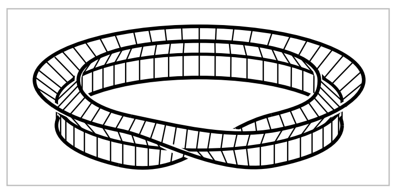

of the Klein bottle is as a square with opposite sides identified via the word $aba^{-1}b$. If one cuts the square along a diagonal and reassembles the resulting two triangles as shown in the figure, one obtains the other representation as a square with sides identified via the word $a^2c^2$. By the proposition, $\pi_1(N_g) \approx \langle a_1, \cdots, a_g \mid a_1^2 \cdots a_g^2 \rangle$. This abelianizes to the direct sum of $\mathbb{Z}_2$ with $g-1$ copies of $\mathbb{Z}$ since in the abelianization we can rechoose the generators to be $a_1, \cdots, a_{g-1}$ and $a_1 + \cdots + a_g$, with $2(a_1 + \cdots + a_g) = 0$. Hence $N_g$ is not homotopy equivalent to $N_h$ if $g \neq h$, nor is $N_g$ homotopy equivalent to any orientable surface $M_h$.

Here is another application of the preceding proposition:

**Corollary 1.28.** *For every group $G$ there is a 2-dimensional cell complex $X_G$ with $\pi_1(X_G) \approx G$.*

**Proof**: Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$. This exists since every group is a quotient of a free group, so the $g_\alpha$'s can be taken to be the generators of this free group with the $r_\beta$'s generators of the kernel of the map from the free group to $G$. Now construct $X_G$ from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells $e^2_\beta$ by the loops specified by the words $r_\beta$. □

**Example 1.29**. If $G = \langle a \mid a^n \rangle = \mathbb{Z}_n$ then $X_G$ is $S^1$ with a cell $e^2$ attached by the map $z \mapsto z^n$, thinking of $S^1$ as the unit circle in $\mathbb{C}$. When $n = 2$ we get $X_G = \mathbb{R}\mathrm{P}^2$, but for $n > 2$ the space $X_G$ is not a surface since there are $n$ 'sheets' of $e^2$ attached at each point of the circle $S^1 \subset X_G$. For example, when $n = 3$ one can construct a neighborhood $N$ of $S^1$ in $X_G$ by taking the product of the graph $\curlyvee$ with the interval $I$, and then identifying the two ends of this product via a one-third twist as shown in the figure. The boundary of $N$ consists of a single circle, formed by the three endpoints of each $\curlyvee$ cross section of $N$. To complete the construction of $X_G$ from $N$ one attaches a disk along the boundary circle of $N$. This cannot be done in $\mathbb{R}^3$, though it can in $\mathbb{R}^4$. For $n = 4$ one would use the graph $\times$ instead of $\curlyvee$, with a one-quarter twist instead of a one-third twist. For larger $n$ one would use an $n$-pointed 'asterisk' and a $1/n$ twist.



## Exercises

**1**. Show that the free product $G * H$ of nontrivial groups $G$ and $H$ has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of $G$ and $H$.

**2**. Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \varnothing$ for all $i, j, k$. Show that $X$ is simply-connected.

**3**. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

**4.** Let $X \subset \mathbb{R}^3$ be the union of $n$ lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

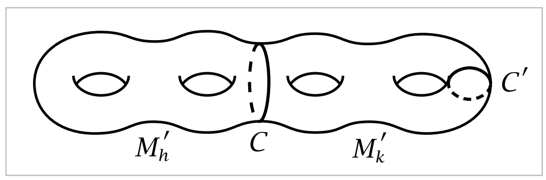
**5.** Let $X \subset \mathbb{R}^2$ be a connected graph that is the union of a finite number of straight line segments. Show that $\pi_1(X)$ is free with a basis consisting of loops formed by the boundaries of the bounded complementary regions of $X$, joined to a basepoint by suitably chosen paths in $X$. [Assume the Jordan curve theorem for polygonal simple closed curves, which is equivalent to the case that $X$ is homeomorphic to $S^1$.]

**6.** Suppose a space $Y$ is obtained from a path-connected subspace $X$ by attaching $n$-cells for a fixed $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism on $\pi_1$. [See the proof of Proposition 1.26.] Apply this to show that the complement of a discrete subspace of $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

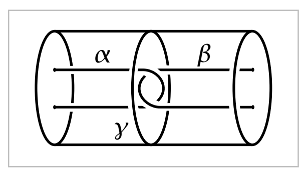
**7.** Let $X$ be the quotient space of $S^2$ obtained by identifying the north and south poles to a single point. Put a cell complex structure on $X$ and use this to compute $\pi_1(X)$.

**8.** Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

**9.** In the surface $M_g$ of genus $g$, let $C$ be a circle that separates $M_g$ into two compact subsurfaces $M_h'$ and $M_k'$ obtained from the closed surfaces $M_h$ and $M_k$ by deleting an open disk from each. Show that $M_h'$ does not retract onto its boundary circle $C$, and hence $M_g$ does not retract onto $C$. [Hint: abelianize $\pi_1$.] But show that $M_g$ does retract onto the nonseparating circle $C'$ in the figure.

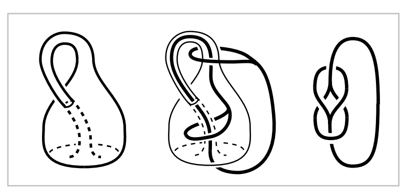



**10.** Consider two arcs $\alpha$ and $\beta$ embedded in $D^2 \times I$ as shown in the figure. The loop $\gamma$ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of $\gamma$ in the complement of $\alpha \cup \beta$.




**11.** The **mapping torus** $T_f$ of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with $f$ basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [One way to do this is to regard $T_f$ as built from $X \vee S^1$ by attaching cells.]

**12.** The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of $X$, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that



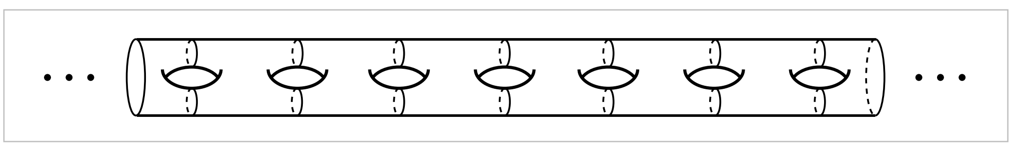
$\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\varepsilon}c^{-1} \rangle$ for $\varepsilon = \pm 1$. (Changing the sign of $\varepsilon$ gives an isomorphic group, as it happens.) Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 - Z)$ for $Z$ the graph shown in the figure. The groups $\pi_1(X)$ and $\pi_1(Y)$ are not isomorphic, but this is not easy to prove; see the discussion in Example 1B.13.

**13.** The space $Y$ in the preceding exercise can be obtained from a disk with two holes by identifying its three boundary circles. There are only two essentially different ways of identifying the three boundary circles. Show that the other way yields a space $Z$ with $\pi_1(Z)$ not isomorphic to $\pi_1(Y)$. [Abelianize the fundamental groups to show they are not isomorphic.]

**14.** Consider the quotient space of a cube $I^3$ obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space $X$ is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

**15.** Given a space $X$ with basepoint $x_0 \in X$, we may construct a CW complex $L(X)$ having a single 0-cell, a 1-cell $e^1_\gamma$ for each loop $\gamma$ in $X$ based at $x_0$, and a 2-cell $e^2_\tau$ for each map $\tau$ of a standard triangle $PQR$ into $X$ taking the three vertices $P$, $Q$, and $R$ of the triangle to $x_0$. The 2-cell $e^2_\tau$ is attached to the three 1-cells that are the loops obtained by restricting $\tau$ to the three oriented edges $PQ$, $PR$, and $QR$. Show that the natural map $L(X) \to X$ induces an isomorphism $\pi_1(L(X)) \approx \pi_1(X, x_0)$.

**16.** Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.



**17.** Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

**18.** In this problem we use the notions of suspension, reduced suspension, cone, and mapping cone defined in Chapter 0. Let $X$ be the subspace of $\mathbb{R}$ consisting of the sequence $1, 1/2, 1/3, 1/4, \cdots$ together with its limit point $0$.

(a) For the suspension $SX$, show that $\pi_1(SX)$ is free on a countably infinite set of generators, and deduce that $\pi_1(SX)$ is countable. In contrast to this, the reduced suspension $\Sigma X$, obtained from $SX$ by collapsing the segment $\{0\} \times I$ to a point, is the shrinking wedge of circles in Example 1.25, with an uncountable fundamental group.

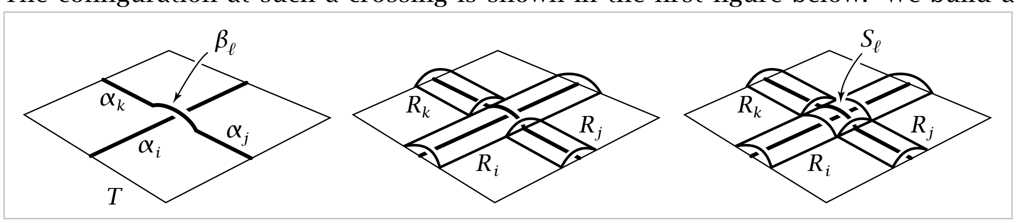
(b) Let $C$ be the mapping cone of the quotient map $SX \to \Sigma X$. Show that $\pi_1(C)$ is uncountable by constructing a homomorphism from $\pi_1(C)$ onto $\prod_\infty \mathbb{Z} / \bigoplus_\infty \mathbb{Z}$. Note that $C$ is the reduced suspension of the cone $CX$. Thus the reduced suspension of a contractible space need not be contractible, unlike the unreduced suspension.

**19**. Show that the subspace of $\mathbb{R}^3$ that is the union of the spheres of radius $1/n$ and center $(1/n, 0, 0)$ for $n = 1, 2, \cdots$ is simply-connected.

**20**. Let $X$ be the subspace of $\mathbb{R}^2$ that is the union of the circles $C_n$ of radius $n$ and center $(n, 0)$ for $n = 1, 2, \cdots$. Show that $\pi_1(X)$ is the free group $*_n \pi_1(C_n)$, the same as for the infinite wedge sum $\bigvee_\infty S^1$. Show that $X$ and $\bigvee_\infty S^1$ are in fact homotopy equivalent, but not homeomorphic.

**21**. Show that the join $X * Y$ of two nonempty spaces $X$ and $Y$ is simply-connected if $X$ is path-connected.

**22**. In this exercise we describe an algorithm for computing a presentation of the fundamental group of the complement of a smooth or piecewise linear knot $K$ in $\mathbb{R}^3$, called the *Wirtinger presentation*. To begin, we position the knot to lie almost flat on a table, so that $K$ consists of finitely many disjoint arcs $\alpha_i$ where it intersects the table top together with finitely many disjoint arcs $\beta_\ell$ where $K$ crosses over itself. The configuration at such a crossing is shown in the first figure below. We build a




2-dimensional complex $X$ that is a deformation retract of $\mathbb{R}^3 - K$ by the following three steps. First, start with the rectangle $T$ formed by the table top. Next, just above each arc $\alpha_i$ place a long, thin rectangular strip $R_i$, curved to run parallel to $\alpha_i$ along the full length of $\alpha_i$ and arched so that the two long edges of $R_i$ are identified with points of $T$, as in the second figure. Any arcs $\beta_\ell$ that cross over $\alpha_i$ are positioned to lie in $R_i$. Finally, over each arc $\beta_\ell$ put a square $S_\ell$, bent downward along its four edges so that these edges are identified with points of three strips $R_i$, $R_j$, and $R_k$ as in the third figure; namely, two opposite edges of $S_\ell$ are identified with short edges of $R_j$ and $R_k$ and the other two opposite edges of $S_\ell$ are identified with two arcs crossing the interior of $R_i$. The knot $K$ is now a subspace of $X$, but after we lift $K$ up slightly into the complement of $X$, it becomes evident that $X$ is a deformation retract of $\mathbb{R}^3 - K$.

(a) Assuming this bit of geometry, show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator $x_i$ for each strip $R_i$ and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square $S_\ell$, where the indices are as in the figures above. [To get the correct signs it is helpful to use an orientation of $K$.]

(b) Use this presentation to show that the abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$.