9
Simplicial Homology

In Chapter 8, we have seen how to take a topological space and assign to it an algebraic object carrying some information about the topology of the space. However, homotopy groups are very hard to calculate. In this chapter we introduce "homology" groups, which can be thought of as a rough approximation to the homotopy groups of a space. In defining the homotopy groups of a space X, we considered all maps $f : S^n \to X$ and ignored those which could be deformed to a constant map. Such a deformation means extending f to a map $D^n \to X$, i.e., filling in the image of S^n under f. So the nth homotopy group could be roughly described as counting those images of S^n in X which *cannot* be filled in. We could think of such images as "n-dimensional holes," so that the annulus has a one-dimensional hole, i.e., a hole which can be bounded by a one-dimensional rope. A 2-sphere would then have a two-dimensional hole, as the hole inside it cannot be bounded by a one-dimensional rope, but has a two-dimensional boundary.

However, this way of understanding $\pi_n(X)$ breaks down when we realize that $\pi_m(S^n)$ is usually not zero if $m > n$, suggesting that the n-sphere usually has lots of m-dimensional holes!

Homology groups offer a different approach to hole counting, and one that behaves slightly more intuitively. For example, with this approach, the n-sphere has one n-dimensional hole and no m-dimensional holes for $m \neq n$ (except $m = 0$ which is an exceptional case).

As with the Euler characteristic, we will begin by defining homology for simplicial complexes before giving a more general definition in Chapter 10. In the simplicial context, a "hole" is some combination of simplices which could

M.D. Crossley, *Essential Topology*, Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84628-194-5_9,

possibly be the boundary of a simplex, or of a combination of simplices, but which is not. For example, the three edges that form the simplicial circle of Example 7.1 look just like the three edges that form the boundary of a 2-simplex. Hence these edges *could* be a boundary, but in the simplicial circle they are not a boundary, since there is no 2-simplex with this collection of edges as its boundary.

9.1 Simplicial Homology Modulo 2

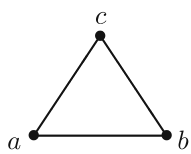
Describing the boundary of a simplex, or combination of simplices, is essential to homology, so we will begin by recalling what we mean by taking the boundary of a simplex, and we will turn this process into an algebraic function.

Recall that a k-simplex $[v_0, \ldots, v_k]$ has $k+1$ faces, each of which is a $(k-1)$-simplex, given by omitting one of the vertices. There is a very useful convention for describing a list in which one item has been omitted – you put a hat over the omitted item. For example, $\hat{a}, b, c$ would denote the list b, c, the letter a having been omitted.

With this convention, if $[v_0, \ldots, v_k]$ is a k-simplex, then its faces can be written as $[v_0, \ldots, \widehat{v_i}, \ldots, v_k]$, where $0 \leq i \leq k$. For example, the faces of a 2-simplex $[a, b, c]$ are

$$[\hat{a}, b, c] = [b, c], \quad [a, \hat{b}, c] = [a, c], \quad [a, b, \hat{c}] = [a, b].$$

The boundary of the k-simplex is the union of these faces, so the boundary of $[a, b, c]$ is the union of $[b, c]$, $[a, c]$ and $[a, b]$, as depicted below.



If, for a given simplicial complex K, we let $S_n(K)$ be the set of all n-simplices in K, then the boundary of each element in $S_n(K)$ is a list of elements of $S_{n-1}(K)$. We would like to construct a "boundary operator" or "boundary function" which takes a simplex and gives its boundary. However, since the boundary is not a single simplex, but a list of simplices, we cannot construct a boundary function $S_n(K) \to S_{n-1}(K)$. There are various ways to get around this, and the quickest way (though not the best, as we will see later) is as follows. Let $C_n(K)$ be the collection of all subsets of $S_n(K)$, so an element of $C_n(K)$ is a subset of $S_n(K)$. Hence, the boundary of an n-simplex is an element of $C_{n-1}(K)$.

Another way of describing $C_n(K)$ is to say that it is the $\mathbf{Z}/2$-vector space spanned by $S_n(K)$, i.e., the set of all linear combinations

$$\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k$$

where, for $1 \leq i \leq k$, λ_i is an element of $\mathbf{Z}/2$, i.e., either 0 or 1, and σ_i is an n-simplex in K. Such a linear combination is called an n-**chain** and, by listing those simplices σ_i which have coefficient 1 in this expression, we get a subset of $S_n(K)$. Hence both descriptions agree, but the second definition has the advantage of bringing us into the realm of linear algebra: We can add two n-chains together to get a third n-chain. Note, however, that we may add two n-chains together and get 0 as the answer. For example, $\sigma + \sigma = 2\sigma = 0$ since the coefficients are added in $\mathbf{Z}/2$.

As the boundary of an n-simplex is an element of $C_{n-1}(K)$, we get, for each $n > 0$, a function $d_n : S_n(K) \to C_{n-1}(K)$. We can extend this to a linear transformation $\delta_n : C_n(K) \to C_{n-1}(K)$ by defining

$$\delta_n(\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k) = \lambda_1 d_n(\sigma_1) + \cdots + \lambda_k d_n(\sigma_k),$$

whenever $\sigma_1, \ldots, \sigma_k$ are n-simplices, and $\lambda_1, \ldots, \lambda_k$ are coefficients in $\mathbf{Z}/2$. This will involve adding up coefficients, which we do modulo 2 of course. For example, if

$$d_n(\sigma_1) = s_1 + s_2 \qquad \text{and} \quad d_n(\sigma_2) = s_2 + s_3,$$

where $s_1, s_2, s_3 \in C_{n-1}(K)$, then

$$\delta_n(\sigma_1 + \sigma_2) = (s_1 + s_2) + (s_2 + s_3) = s_1 + s_3.$$

So, for each $n \geq 0$, we have a $\mathbf{Z}/2$-vector space $C_n(K)$ and, for each $n \geq 1$, a linear transformation

$$\delta_n : C_n(K) \longrightarrow C_{n-1}(K)$$

called the **boundary operator**.

For example, applying this to a 1-simplex $[a, b]$ gives

$$\delta_1[a, b] = a + b,$$

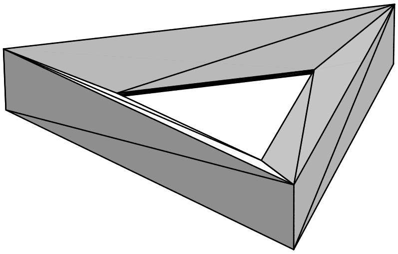
and applying it to a 2-simplex $[a, b, c]$, we get

$$\delta_2[a, b, c] = [b, c] + [a, c] + [a, b].$$

In general, the effect of applying δ_n to an arbitrary n-simplex is given by the formula

$$\delta_n[v_0, \ldots, v_n] = \sum_{i=0}^{n} [v_0, \ldots, \widehat{v_i}, \ldots, v_n].$$

The aim of homology is to study those combinations of simplices which *could* be boundaries of simplices (or of a combination of simplices) but which are not. For example, the three innermost edges of the simplicial torus



look just like the boundary of a 2-simplex. But there is no 2-simplex in this simplicial torus whose boundary is that combination of edges. In that sense, these three edges *could* be a boundary, but are not.

The problem is to determine which combinations of simplices could be boundaries. Obviously, the combination must look something like the boundary of a simplex, so we will resolve this problem by studying boundaries of simplices.

As we have seen, the boundary of an n-simplex is a union of $(n-1)$-simplices, and these $(n-1)$-simplices cannot be arbitrary: They are related to each other by the condition that each face of one of these $(n-1)$-simplices is a face of exactly one other $(n-1)$-simplex. For example, the boundary of the 2-simplex $[a,b,c]$ consists of the simplices $[b,c]$, $[a,c]$, $[a,b]$. The faces of these boundary simplices are: b, c, a, c and a, b. Each element in this list occurs twice as each face occurs in two boundary simplices. Putting that another way, $\delta_2[a,b,c] = [b,c] + [a,c] + [a,b]$, and

$$\delta_1(\delta_2[a,b,c]) = \delta_1([b,c]+[a,c]+[a,b]) = (b+c)+(a+c)+(a+b) = 2a+2b+2c.$$

But, since we are working modulo 2, this is 0. In other words $\delta_1 \circ \delta_2$ is zero on each 2-simplex. This generalizes as follows.

Lemma 9.1

For every $n \geq 1$, the composite

$$\delta_n \circ \delta_{n+1} : C_{n+1}(K) \longrightarrow C_{n-1}(K)$$

is the zero linear transformation.

Proof

Certainly this composite will be a linear transformation, so it is enough to verify it for every element of a basis for $C_{n+1}(K)$. By definition, one basis is $S_{n+1}(K)$, so we will show that $\delta_n(\delta_{n+1}(\sigma)) = 0$ for every $(n+1)$-simplex σ.

Let $\sigma = [v_0, \dots, v_{n+1}]$, so

$$\delta_{n+1}(\sigma) = \sum_{i=0}^{n+1} [v_0, \dots, \widehat{v_i}, \dots, v_{n+1}].$$

Then

$$\delta_n \delta_{n+1}(\sigma) = \sum_{\substack{j=0 \\ j \neq i}}^{n+1} \sum_{i=0}^{n+1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_{n+1}].$$

But every summand here occurs twice: the simplex $[v_0, \dots, \widehat{v_a}, \dots, \widehat{v_b}, \dots, v_{n+1}]$ occurs when $i = a$ and $j = b$, and also when $i = b$ and $j = a$. Since we are working modulo 2, this means that all the summands cancel, leaving

$$\delta_n \delta_{n+1}(\sigma) = 0. \qquad \square$$

Lemma 9.1 shows that we have a sequence of $\mathbf{Z}/2$-vector spaces and linear transformations

$$\cdots \longrightarrow C_n(K) \xrightarrow{\delta_n} C_{n-1}(K) \xrightarrow{\delta_{n-1}} C_{n-2}(K) \longrightarrow \cdots \longrightarrow C_1(K) \xrightarrow{\delta_1} C_0(K),$$

where the composite of any two transformations is 0. Such a sequence is called a **chain complex** and is denoted by $(C_*(K), \delta_*)$, or $C_*(K)$, or even C_* according to context.

So the boundary of an $n+1$-simplex is in the kernel of δ_n. Traditionally, this is expressed by saying that "a boundary has no boundary." This suggests a way of detecting combinations of simplices which "could" be boundaries: They are those combinations which have no boundary themselves. In other words, they are elements of Ker δ_n, and for $n > 0$, we write $Z_n(K)$ for Ker δ_n, abbreviating this to Z_n if the context makes it clear which simplicial complex we are considering. Elements of Z_n are called **cycles**, the Z coming from the German word for "cycle." For convenience, we define $Z_0(K)$ to be $C_0(K)$.

However, we wish to discard those combinations of simplices which actually are boundaries. These are easily recognised: they are the elements of Im δ_{n+1}. We write $B_n(K)$ (or just B_n) for Im δ_{n+1} and refer to elements of B_n as **boundaries**.

Lemma 9.1 says that $B_n \subset Z_n$ for all $n \geq 0$. So one way to "discard" the cycles which are actually boundaries is to take the **quotient group** Z_n/B_n, whose elements are equivalence classes of cycles under the relation $z_1 \sim z_2$ if $z_1 - z_2$ is in B_n.

Definition: The nth **homology group** of a simplicial complex K is the quotient

$$H_n(K) = \frac{Z_n(K)}{B_n(K)} = \begin{cases} \text{Ker } \delta_n/\text{Im } \delta_{n+1} & \text{if} \quad n > 0, \\ C_0/\text{Im } \delta_1 & \text{if} \quad n = 0. \end{cases}$$

The **homology** of K is the collection

$$H_*(K) = \{H_0(K), H_1(K), H_2(K), \cdots\}.$$

The group $H_n(K)$ is sometimes referred to as the **degree** n part, or **dimension** n part, of the homology of K. Note that since $Z_n(K)$ and $B_n(K)$ are $\mathbf{Z}/2$-vector spaces, with $B_n(K)$ a vector subspace of $Z_n(K)$, $H_n(K)$ is, in fact, a $\mathbf{Z}/2$-vector space, not just a group. However, the term "homology group" is traditional as there is a more common construction, integral homology, which we will meet shortly, which yields an Abelian group, not a vector space.

Since the elements of $H_n(K)$ are equivalence classes of cycles modulo boundaries, we say that two chains z_1 and z_2 are **homologous** if their difference $z_1 - z_2$ is a boundary, i.e., $z_1 - z_2 \in B_n$.

Example 9.2

Let K be the simplicial circle of Example 7.1. This has three 0-simplices, which we label a, b, c, and three 1-simplices, $[a,b]$, $[b,c]$, $[a,c]$. So C_0 and C_1 both have dimension 3, while $C_i = 0$ if $i > 1$. Thus there is only one map in the chain complex which can possibly be non-zero, namely δ_1. So the only interesting part of the chain complex is

$$C_1 \xrightarrow{\delta_1} C_0,$$

where $\delta_1[a,b] = a+b$, $\delta_1[b,c] = b+c$, $\delta_1[a,c] = a+c$. Let $\sigma = \lambda_1[a,b] + \lambda_2[b,c] + \lambda_3[a,c]$ be an arbitrary element of C_1. Then

$$\delta_1(\sigma) = \lambda_1(a+b) + \lambda_2(b+c) + \lambda_3(a+c) = (\lambda_1+\lambda_3)a + (\lambda_1+\lambda_2)b + (\lambda_2+\lambda_3)c.$$

If $\delta_1(\sigma) = 0$, then $\lambda_1 + \lambda_3 = 0$, i.e., $\lambda_3 = \lambda_1$ (since we are working over $\mathbf{Z}/2$, so $-1 = +1$), and $\lambda_1 + \lambda_2 = 0$, i.e., $\lambda_1 = \lambda_2$. In other words, if $\sigma \in \text{Ker } \delta_1$, then

$$\sigma = \lambda([a,b] + [b,c] + [a,c])$$

for some $\lambda \in \mathbf{Z}/2$. Hence $\dim Z_1 = \dim \text{Ker } \delta_1 = 1$. The map δ_2 is 0, so its image B_1 is 0, so $H_1(K) = Z_1/B_1 = Z_1 = \mathbf{Z}/2$.

By the rank-and-nullity theorem of linear algebra, $\dim \text{Im } \delta_1 = \dim C_1 - \dim \text{Ker } \delta_1 = 3 - 1 = 2$. Hence $\dim H_0(K) = \dim(C_0/B_0) = \dim(C_0) - \dim(B_0) = 3 - 2 = 1$.

As $C_i = 0$ for $i > 1$, so $H_i(K) = 0$ for $i > 1$, and we have a complete calculation of the homology groups: $H_0(K) = \mathbf{Z}/2$, $H_1(K) = \mathbf{Z}/2$ and $H_i(K) = 0$ for $i > 1$.

Example 9.3

The simplicial square of Example 7.2 has four 0-simplices, five 1-simplices and two 2-simplices, so the non-zero part of the chain complex is longer:

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0.$$

As in Example 9.2, the image of δ_1 consists of all sums $v_i + v_j$ of two vertices, the sum $v_1 + v_2 + v_3 + v_4$ of all four vertices, and 0. So $B_0 = \operatorname{Im} \delta_1$ contains 8 elements, and $\dim \operatorname{Im} \delta_1$ must be 3. Hence $\dim H_0 = \dim(C_0) - \dim(B_0) = 4 - 3 = 1$.

As $\dim \operatorname{Im} \delta_1 = 3$, the rank-and-nullity theorem shows that $\dim \operatorname{Ker} \delta_1 = 2$. Since the two 2-simplices have different boundaries, so $\dim \operatorname{Im} \delta_2 = 2$, and hence $H_1 = 0$.

Finally, since $\dim \operatorname{Ker} \delta_2 = 0$, we see that $H_2 = 0$ as well. In other words, the square has non-zero homology only in dimension 0. Hence $H_0 = \mathbf{Z}/2$, $H_i = 0$ for $i > 0$.

In this situation, where the only non-zero homology group is in dimension 0, the simplicial complex is said to be **acyclic**.

Example 9.4

For the simplicial torus of Example 7.8, the non-zero part of the chain complex is

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0,$$

where $\dim C_2 = 18$, $\dim C_1 = 27$, $\dim C_0 = 9$.

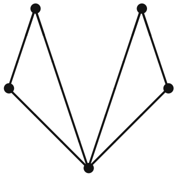
As usual, the image of δ_1 consists of all sums of an even number of vertices, so this has dimension $\dim C_0 - 1$, i.e., 8, so $\dim H_0 = 1$.

As $\dim \operatorname{Im} \delta_1 = 8$, we see that $\dim \operatorname{Ker} \delta_1 = 19$. Now, if we take all the 2-simplices in the torus, their boundaries will add up to 0, since each edge occurs as the face of exactly two 2-simplices. Hence this is an element of $\operatorname{Ker} \delta_2$. Moreover, an easy computation shows that it is the only non-zero 2-cycle: For if the 2-simplex σ is a summand in a given 2-cycle, then this cycle must also include every 2-simplex which shares an edge with σ. And then the cycle must include every 2-simplex which shares an edge with any of these simplices. Carrying on, we see that every 2-simplex must be in the cycle. Thus $\dim \operatorname{Ker} \delta_2 = 1$, which implies that $\dim \operatorname{Im} \delta_2 = 17$. Hence $\dim H_1 = 2$, and $\dim H_2 = 1$.

Hence $H_0 = \mathbf{Z}/2$, $H_1 = \mathbf{Z}/2 \oplus \mathbf{Z}/2$, $H_2 = \mathbf{Z}/2$, $H_i = 0$ for $i > 2$.

Example 9.5

A pair of rabbit ears (i.e., a figure of eight)



has six 1-cells and five 0-cells, so its chain complex is

$$C_1 \to C_0.$$

As usual, $\dim \mathrm{Im}\ \delta_1 = \dim C_0 - 1 = 4$, so $\dim H_0 = 1$ and $\dim H_1 = 2$. Hence $H_0 = \mathbf{Z}/2$, $H_1 = \mathbf{Z}/2 \oplus \mathbf{Z}/2$, $H_i = 0$ for $i > 1$.

Example 9.6

We can produce a simplicial sphere by taking a tetrahedron, as in Example 7.11. This has four 0-simplices, six 1-simplices (consisting of all possible pairs of vertices; note that $\binom{4}{2} = 6$) and four 2-simplices (consisting of all possible triples of vertices; $\binom{4}{3} = 4$).

As usual, $\dim H_0 = 1$, since Im δ_1 consists of all sums of two vertices, so has dimension 3. This tells us that $\dim \mathrm{Ker}\ \delta_1 = 6 - 3 = 3$. Now δ_2 is almost injective: Each 1-simplex on the boundary of a given 2-simplex is shared with only one other 2-simplex, and the only element of C_2 which is in Ker δ_2 is the sum of all four 2-simplices. Hence $\dim \mathrm{Ker}\ \delta_2 = 1$, and $\dim \mathrm{Im}\ \delta_2 = 3$.

Hence $H_0 = \mathbf{Z}/2$, $H_1 = 0$, $H_2 = \mathbf{Z}/2$ and $H_i = 0$ for $i > 2$.

Notice that in all of these examples $H_0 = \mathbf{Z}/2$. This is a particular case of the following general fact about H_0.

Proposition 9.7

For any simplicial complex K, the dimension of $H_0(K)$ is equal to the number of path components in K, i.e., $\dim H_0$ is the number of elements of $\pi_0(K)$.

Proof

The group $\pi_0(K)$ is the set of pointed maps $S^0 \to K$ modulo homotopy. Every pointed map $f : S^0 \to K$ is determined by the point $f(-1)$ in K, so $\pi_0(K)$ is equivalent to the set K modulo the relation that $x \sim y$ if, and only if, there is a continuous path $[0,1] \to K$ that takes the values x at 0 and y at 1. Each point $x \in K$ is in the interior of exactly one simplex, and is joined to each vertex of that simplex by a continuous path. So $\pi_0(K)$ is equivalent to the set S_0 of 0-simplices of K, modulo path-connectivity.

On the other hand, $H_0(K) = C_0/\text{Im } \delta_1$ is the set of $\mathbf{Z}/2$-linear combinations of vertices of K modulo the relation that two vertices, x, y are equivalent, if $y - x$ is in the image of δ_1, i.e., there is a list of 1-simplices $e_1, \ldots, e_n$ with $\delta_1(e_1 + \cdots + e_n) = y - x$. This condition means that, if we arrange $e_1, \ldots, e_n$ appropriately, e_1 is a 1-simplex from x to another vertex, x_1, e_2 is a 1-simplex from x_1 to another vertex x_2, and so on, up to e_n which is a 1-simplex from x_{n-1} to y.

If we choose a set of vertices $\{v_1, \ldots, v_m\}$ such that every vertex is equivalent to one, and only one, of these vertices under this δ_1 relation, then every element of $H_0(K)$ can be represented uniquely by a $\mathbf{Z}/2$-linear combination of $\{v_1, \ldots, v_m\}$. Hence $\dim H_0(K) = m$.

Thus a basis for $H_0(K)$ is obtained by taking the vertices of K and applying one equivalence relation, and $\pi_0(K)$ can be computed by taking the vertices of K and applying *another* equivalence relation. We will complete the proof by showing that these equivalence relations actually coincide.

If $f : [0,1] \to K$ is a path from one vertex of K to another then we can replace f by another path (in fact, one that is homotopic to f) whose image is contained in the union of the 1-simplices of K. If we take precisely those 1-simplices that form the image of this replacement path, then that forms an element of C_1 whose boundary is $f(1) - f(0)$. Hence if x and y correspond to equivalent elements of $\pi_0(K)$, then $y - x \in \text{Im } \delta_1$.

And vice versa: if $y - x \in \text{Im } \delta_1$ in H_0 then there is a list of 1-simplices whose boundary is $y - x$ and which, consequently, can be put together to form a path from x to y. Thus the two equivalence relations are the same, completing the proof of the Proposition. □

Example 9.8

In the simplicial complex

there are two path components, and the chain complex for simplicial homology is

$$C_1 \to C_0$$

where $\dim C_0 = 3$, $\dim C_1 = 1$. The boundary of the single 1-simplex is non-zero, so $\dim \text{Im } \delta_1 = 1$, and so $\dim H_0 = 2$, while $\dim H_1 = 0$.

The Euler number is closely related to simplicial homology:

Proposition 9.9

If K is a simplicial complex, then

$$\chi(K) = \sum_{n \geq 0} (-1)^n \dim H_n.$$

Proof

The definition of the Euler number is $\chi(K) = \sum_{n \geq 0} (-1)^n \dim C_n(K)$. Since $H_n = \text{Ker } \delta_n / \text{Im } \delta_{n+1}$, its dimension is given by

$$\dim H_n = (\dim C_n - \dim \text{Im } \delta_n) - \dim \text{Im } \delta_{n+1}.$$

In forming the alternating sum, the last two terms on the right cancel, giving

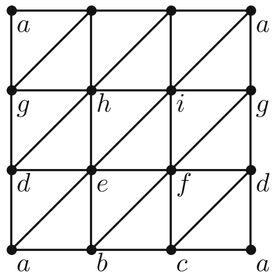
$$\sum_{n \geq 0} (-1)^n \dim H_n = \sum_{n \geq 0} (-1)^n \dim C_n = \chi(K). \qquad \square$$

9.2 Limitations of Homology Modulo 2

All the preceding examples, the circle, square, annulus, sphere, torus and rabbit ears, have different homology except for the circle and annulus, which both have the same homology. Since the circle and annulus are homotopy equivalent, this is no great surprise. But it suggests that homology is very powerful as it can distinguish all the other examples we have computed.

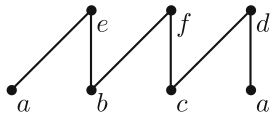
Nevertheless, it is not as powerful as it could be. For example, the Klein bottle can be triangulated in a similar way to the torus, and the resulting homology groups will be the same as for the torus, with $\dim H_0 = 1$, $\dim H_1 = 2$, $\dim H_2 = 1$ and $\dim H_i = 0$ for $i > 2$. Yet the Klein bottle and the torus are not homotopy equivalent.

By looking at the torus and Klein bottle more closely we can see how to improve the homology theory that we have. To understand the similarities and differences, we imagine the triangulation of the torus as coming from a triangulation of the square pictured below. In order to get the torus from folding up this square, we would have to stretch some edges and squeeze others, but we will end up with the triangulation pictured in Example 7.8.



The vertex labels reflect the way that several vertices are identified with each other when we form the torus. For example, all four corners are labelled a, as they are all glued together when we form the torus. If we glue this square together to get the Klein bottle, in the way described by Example 5.57, then again, all four corners would be identified together, and the vertex labelling above would be correct for the Klein bottle. It is for this reason that the two vertices in the middle of the top row have not been labelled, as their labels change according to whether we form the torus or the Klein bottle. For the torus they should be labelled b, c, from left to right, whereas for the Klein bottle they should be labelled c, b.

If we take the chain $[a, b] + [b, c] + [c, a]$ corresponding to the bottom row, then we see that this is in Ker δ_1 as its boundary is $a+b+b+c+c+a = 0$. So this gives rise to some class in homology. And if we add anything from Im δ_2, we get the same class in homology. Well, suppose we add the boundaries of the 2-simplices $[a, e, b]$, $[b, f, c]$, $[c, d, a]$. This gives the longer chain $[a, e] + [e, b] + [b, f] + [f, c] + [c, d] + [d, a]$, corresponding to the zig-zag chain



Then, adding the boundaries of the 2-simplices $[a, d, e]$, $[b, e, f]$, $[c, d, f]$, we get the chain

$$[a, d]+[d, e]+[e, b]+[e, b]+[e, f]+[f, c]+[c, f]+[f, d]+[d, a] = [d, e]+[e, f]+[f, d],$$

i.e., the second row. Carrying on in the same way, we see that $[a, b]+[b, c]+[c, a]$ gives the same homology class as the third row $[g, h] + [h, i] + [i, g]$. And, for the torus, if we move up one more row, we get back to $[a, b] + [b, c] + [c, a]$. In the Klein bottle, however, we get $[a, c] + [c, b] + [b, a]$, i.e., the same simplices, but each in the opposite direction.

As we have formulated homology, this is just the same as $[a, b]+[b, c]+[c, a]$, but if we could incorporate direction in some way, then we might be able to distinguish between these two chains and, hence, between the torus and the

Klein bottle. In particular, if we could make $[b, a]$ equal to $-[a, b]$, then, for the Klein bottle, we would have

$$\begin{aligned}[a, b] + [b, c] + [c, a] \equiv [a, c] + [c, b] + [b, a] &= -[c, a] - [b, c] - [a, b] \\ &= -([a, b] + [b, c] + [c, a]) \bmod \text{Im } \delta_2.\end{aligned}$$

In other words, $2([a, b] + [b, c] + [c, a]) \in \text{Im } \delta_2$, so we would have some non-zero element of Im δ_2, whereas in the torus, no such element arises.

Clearly, in order to distinguish $[a, b]$ from $-[a, b]$, we need to move away from $\mathbf{Z}/2$ and work, instead, with integer coefficients. We also need to establish what the analogues of $[b, a] = -[a, b]$ are for 2-simplices and higher-dimension simplices.

For a 2-simplex $[a, b, c]$, there are six different ways of ordering the vertices: $[a, b, c]$, $[b, c, a]$, $[c, a, b]$, $[b, a, c]$, $[a, c, b]$, $[c, b, a]$. Each of these can be turned into any of the others by swapping pairs of vertices repeatedly. In particular, any can be turned into $[a, b, c]$ by swapping pairs of vertices. Some require just one swap: $[b, a, c]$, $[c, b, a]$, $[a, c, b]$ while some require two swaps: $[b, c, a]$, $[c, a, b]$. With more vertices, even more swaps are necessary.

We group $[b, a, c]$, $[c, b, a]$ and $[a, c, b]$ together, being the ones which need an odd number of swaps. And we group $[a, b, c]$, $[b, c, a]$ and $[c, a, b]$ together, being those which need an even number of swaps. We then consider $[b, a, c]$, $[c, b, a]$ and $[a, c, b]$ to be the same **oriented simplex**, and we consider $[a, b, c]$, $[b, c, a]$, $[c, a, b]$ to be the same as each other, but "opposite" to $[b, a, c]$, $[c, b, a]$ and $[a, c, b]$. When working with chains of simplices, we will insist that $[b, a, c] = -[a, b, c]$.

In general, an n-simplex has $(n + 1)!$ orderings, but each can be turned into any other by a number of swaps. We consider two orderings to have the same orientation if they differ by an even number of swaps, and to have the opposite orientation if they differ by an odd number of swaps. An **oriented simplex** is then a set of orderings of the vertex list of a simplex, such that all these orderings have the same orientation, and any other ordering having the same orientation is in the set.

9.3 Integral Simplicial Homology

Given a simplicial complex K, we let $S_n(K)$ be the set of all n-simplices as before. But now we let $C_n(K)$ be the set of all $\mathbf{Z}$-linear combinations of oriented simplices, subject to the relation that if σ is an oriented simplex, then $(-1)\sigma$ is the same simplex with the opposite orientation. So, $C_n(K)$ consists of sums such as

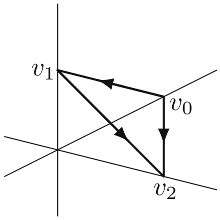
$$2\sigma_1 + 3\sigma_2 - 2\sigma_4,$$

and if τ is the same simplex as σ_4 but with its vertices changed by an odd permutation, then this element of $C_n(K)$ is the same as

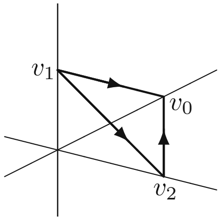
$$2\sigma_1 + 3\sigma_2 + 2\tau.$$

As before, an element of $C_n(K)$ is called an n-**chain**.

Now, we have seen that the boundary of a k-simplex consists of the $k+1$ different $(k-1)$-simplices obtained by omitting one of the vertices. Hence the boundary of $[v_0, v_1, v_2]$ consists of $[v_1, v_2]$, $[v_0, v_2]$ and $[v_0, v_1]$. If we imagine each of these 1-simplices as being given an arrow from the first listed vertex to the second, then the arrows on the boundary of the 2-simplex are as follows:



Instinctively we know that the arrow on $[v_0, v_2]$ should point the other way, so that all the arrows would be anti-clockwise. This instinct is borne out by comparing this picture with the corresponding picture for $[v_1, v_2, v_0]$. This is the same oriented simplex as $[v_0, v_1, v_2]$, yet if we orient the boundary simplices $[v_2, v_0]$, $[v_1, v_0]$, $[v_1, v_2]$ by giving each an arrow from the first vertex to the second, as above, then we get the following picture:



The direction of two of the arrows disagrees with the earlier picture. Since this is the boundary of the same oriented simplex, they should look the same.

The solution is to reverse the orientation on the second simplex in each case, so that the boundary of $[v_0, v_1, v_2]$ is $[v_1, v_2]$, $-[v_0, v_2]$, $[v_0, v_1]$ and the boundary of $[v_1, v_2, v_0]$ is $[v_2, v_0]$, $-[v_1, v_0]$ and $[v_1, v_2]$. Since $-[a, b] = [b, a]$, we see that these two sets of oriented simplices are the same.

Similarly, in calculating the boundary of an arbitrary simplex $[v_0, \ldots, v_n]$ we would need to negate any simplex $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ on the boundary where i is odd. This leads us to define the boundary operator $\delta_n : C_n(K) \to C_{n-1}(K)$

by

$$\delta_n[v_0, \ldots, v_n] = \sum_{i=0}^{n}(-1)^i[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$$

for a simplex $[v_0, \ldots, v_n]$, and $\delta_n(\lambda x + \mu y) = \lambda\delta_n(x) + \mu\delta_n(y)$ for any two elements $x, y \in C_n(K)$ and coefficients $\lambda, \mu \in \mathbf{Z}$. For example,

$$\delta_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

as above, and

$$\delta_1[v_0, v_1] = v_1 - v_0.$$

Note that if we swap a pair of vertices in $[v_0, \ldots, v_n]$ then $\delta_n[v_0, \ldots, v_n]$ changes sign, i.e., δ_n respects the orientation. For example,

$$\delta_2[v_0, v_2, v_1] = [v_2, v_1] - [v_0, v_1] + [v_0, v_2] = -([v_1, v_2] + [v_2, v_0] + [v_0, v_1]).$$

From these examples, we can see that

$$\begin{aligned}(\delta_1 \circ \delta_2)[v_0, v_1, v_2] &= \delta_1[v_1, v_2] - \delta_1[v_0, v_2] + \delta_1[v_0, v_1] \\ &= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0,\end{aligned}$$

i.e., $\delta_1 \circ \delta_2$ is zero on any 2-simplex. This generalizes to the following analogue of Lemma 9.1 which is proved in just the same way as that result.

Lemma 9.10

For every $n \geq 1$, the composite

$$\delta_n \circ \delta_{n+1} : C_{n+1} \longrightarrow C_{n-1}$$

is the zero homomorphism.

Thus we have a chain complex as before. So we define $B_n = \text{Im } \delta_n$ and $Z_n = \text{Ker } \delta_{n-1}$ (unless $n = 0$, in which case we set $Z_0 = C_0$), referring to elements of B_n as boundaries and elements of Z_n as cycles. The lemma assures us that $B_n \subset Z_n$, so we can define the nth **integral homology group** of the complex K to be the quotient $H_n(K) = Z_n/B_n$. For consistency, we will now use the standard terminology $H_n(K; \mathbf{Z}/2)$ to denote the **mod 2 homology** of K, i.e., the homology built around $\mathbf{Z}/2$ that we introduced in Section 9.1. To emphasize the difference, we sometimes write $H_n(K; \mathbf{Z})$ for integral homology.

Example 9.11

Let K be the simplicial circle of Example 7.1, with vertices v_0, v_1, v_2. If we choose orientations for the 1-simplices as follows: $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_0]$, then

$$\begin{aligned}\delta_1(\lambda_0[v_0, v_1] + \lambda_1[v_1, v_2] + &\lambda_2[v_2, v_0]) \\ &= \lambda_0(v_1 - v_0) + \lambda_1(v_2 - v_1) + \lambda_2(v_0 - v_2) \\ &= (\lambda_2 - \lambda_0)v_0 + (\lambda_0 - \lambda_1)v_1 + (\lambda_1 - \lambda_2)v_2.\end{aligned}$$

If this is 0, then $\lambda_0 = \lambda_1 = \lambda_2$. Hence Ker δ_1 consists of all integer multiples of $[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$ and is isomorphic to $\mathbf{Z}$. As $\delta_2 = 0$, we see that $H_1(K) = \mathbf{Z}$.

The calculation of δ_1 above also shows that the image of δ_1 consists of all expressions $\mu_0 v_0 + \mu_1 v_1 + \mu_2 v_2$ where $\mu_2 = -(\mu_0 + \mu_1)$. Hence this image is generated by $v_0 - v_2$, $v_1 - v_2$ and is isomorphic to $\mathbf{Z}^2$. If $a_0 v_0 + a_1 v_1 + a_2 v_2$ is an arbitrary element of C_0, then we can express it as $a_0(v_0 - v_2) + a_1(v_1 - v_2) + (a_2 + a_1 + a_0)v_2$, i.e., as an element of Im δ_1 plus some multiple of v_2. Hence the quotient group C_0/B_0 is generated by v_2 and is isomorphic to $\mathbf{Z}$.

Thus $H_0(K) = \mathbf{Z}$, $H_1(K) = \mathbf{Z}$ and $H_i(K) = 0$ for $i > 1$.

Note that it is no longer appropriate to give the dimensions of the nth homology groups, because they are not vector spaces, but Abelian groups, and they need not be free. This is partly what makes integral homology more informative: It cannot be reduced to a list of dimensions in the same way as $\mathbf{Z}/2$ homology.

For most of the examples that we have met, the integral homology looks much like the $\mathbf{Z}/2$ homology (but with $\mathbf{Z}/2$ replaced by $\mathbf{Z}$):

Example 9.12

If K is the simplicial square of Example 7.2, then $H_0(K) = \mathbf{Z}$ and $H_i(K) = 0$ for $i > 0$.

Example 9.13

If K is the simplicial annulus of Example 7.3, then $H_0(K) = \mathbf{Z}$, $H_1(K) = \mathbf{Z}$ and $H_i(K) = 0$ for $i > 1$.

Example 9.14

If K is the simplicial sphere, then $H_0(K) = \mathbf{Z}$, $H_2(K) = \mathbf{Z}$ and $H_i(K) = 0$ otherwise.

Example 9.15

If K is the simplicial torus, then $H_0(K) = \mathbf{Z}$, $H_1(K) = \mathbf{Z} \oplus \mathbf{Z}$, $H_2(K) = \mathbf{Z}$ and $H_i(K) = 0$ for $i > 2$.

Note that in these examples $H_0 = \mathbf{Z}$. This is an example of the integral analogue of Proposition 9.7.

Proposition 9.16

The group $H_0(K)$ is a free Abelian group whose rank is equal to the number of path components in K.

This can be proved in exactly the same way as in the mod 2 case.

Example 9.17

If K is the Klein bottle, then we have seen that $H_0(K) = \mathbf{Z}$, as the simplicial complex is connected. Unlike the torus, $H_2(K)$ is now zero, as Ker $\delta_2 = 0$. To see why this is so, note that every 1-simplex occurs as a face of exactly two 2-simplices. For example, $[a, b]$ occurs as a face of $[a, b, e]$ and $[a, b, i]$. So if $[a, b, e]$ is a summand in a chain in Ker δ_2, then so must $[a, b, i]$ be. Moreover, the coefficient of $[a, b, e]$ determines the coefficient of $[a, b, i]$. You can quickly check that if $[a, b, i]$ is a summand in the chain, then so is $[a, i, g]$, because of their common boundary simplex $[a, i]$. And, pursuing this reasoning, you will quickly find that every 2-simplex must occur in the chain. In fact, if we order the simplices in the right way, for example anticlockwise in the earlier diagram, they must all have the same coefficient. This shows that the only possible elements of Ker δ_2 are multiples of this sum of all the 2-simplices. Then a quick calculation reveals that δ_2 is not actually zero on this sum, but is equal to $2([a, b] + [b, c] + [c, a])$. Hence Ker $\delta_2 = 0$, so $H_2(K) = 0$, in contrast to the torus.

As $H_0(K) = \mathbf{Z}$, we know that Im δ_1 is a free Abelian group of rank 8. Hence Ker δ_1 is a free Abelian group of rank $27 - 8 = 19$. Since Ker $\delta_2 = 0$, we know that Im $\delta_2 \approx C_2$ is a free Abelian group of rank 18. However, whereas the chain $[a, b] + [b, c] + [c, a]$ belongs to Ker δ_1, only $2([a, b] + [b, c] + [c, a])$ is in Im δ_2. Hence $H_1(K) = \mathbf{Z} \oplus \mathbf{Z}/2$, again differing slightly from the torus.

In all our other calculations of integral homology, the homology groups turned out to be direct sums of copies of $\mathbf{Z}$. In this example, we see a finite summand $\mathbf{Z}/2$ appearing. Such finite subgroups of integral homology groups are called **torsion** because they tend to arise from the sort of "twisting" that produces a Klein bottle instead of a torus. So, for example, we would say that the homology of a torus is "torsion-free", whereas the homology of the Klein bottle "has torsion."

The close connection between integral and $\mathbf{Z}/2$-homology in the other examples is not a coincidence, and there is a connection between the two which even explains the Klein bottle's $\mathbf{Z}/2$-homology. This is part of a general result called the universal coefficient theorem, a proof of which can be found in Section 3.A of [5].

Theorem 9.18 (Universal Coefficient Theorem for $\mathbf{Z}/2$)

For any simplicial complex K,

$$H_n(K;\mathbf{Z}/2) = (H_n(K)\otimes \mathbf{Z}/2) \oplus \mathrm{Tor}(H_{n-1}(K),\mathbf{Z}/2).$$

The **tensor product** operation $\otimes$ and the Tor operator may be unfamiliar, but it is easy to desribe their action, at least for finitely generated groups. Every finitely generated Abelian group G can be expressed as a direct sum

$$G = \mathbf{Z}^n \oplus \mathbf{Z}/p_1^{r_1} \oplus \mathbf{Z}/p_2^{r_2} \oplus \cdots \oplus \mathbf{Z}/p_m^{r_m}$$

for some non-negative integers $n, r_1, r_2, \ldots, r_m$ and primes $p_1, \ldots, p_m$. Both Tor and $\otimes$ respect this direct sum, and act on the individual summands according to the following rules:

$$\begin{array}{lll} \mathbf{Z}\otimes\mathbf{Z}/2 = \mathbf{Z}/2, & \mathbf{Z}/2^r\otimes\mathbf{Z}/2 = \mathbf{Z}/2 & \mathbf{Z}/p^r\otimes\mathbf{Z}/2, = 0 \text{ if } p \text{ is odd},\\ \mathrm{Tor}(\mathbf{Z},\mathbf{Z}/2) = 0, & \mathrm{Tor}(\mathbf{Z}/2^r,\mathbf{Z}/2) = \mathbf{Z}/2, & \mathrm{Tor}(\mathbf{Z}/p^r,\mathbf{Z}/2) = 0 \text{ if } p \text{ is odd}. \end{array}$$

Thus, for example, $(\mathbf{Z}\oplus\mathbf{Z}/4\oplus\mathbf{Z}/3)\otimes\mathbf{Z}/2 = \mathbf{Z}/2\oplus\mathbf{Z}/2$, whereas $\mathrm{Tor}(\mathbf{Z}\oplus\mathbf{Z}/4\oplus\mathbf{Z}/3,\mathbf{Z}/2) = \mathbf{Z}/2$.

Example 9.19

Given that the integral homology of the Klein bottle is

$$H_i(K) = \begin{cases} \mathbf{Z} & \text{if } i = 0,\\ \mathbf{Z}\oplus\mathbf{Z}/2 & \text{if } i = 1,\\ 0 & \text{if } i > 1, \end{cases}$$

the universal coefficient theorem then tells us that

$$\begin{aligned} H_0(K;\mathbf{Z}/2) &= \mathbf{Z}\otimes\mathbf{Z}/2 = \mathbf{Z}/2,\\ H_1(K;\mathbf{Z}/2) &= (\mathbf{Z}\oplus\mathbf{Z}/2)\otimes\mathbf{Z}/2\oplus\mathrm{Tor}(\mathbf{Z},\mathbf{Z}/2) = \mathbf{Z}/2\oplus\mathbf{Z}/2,\\ H_2(K;\mathbf{Z}/2) &= (0\otimes\mathbf{Z}/2)\oplus\mathrm{Tor}(\mathbf{Z}\oplus\mathbf{Z}/2,\mathbf{Z}/2) = \mathbf{Z}/2,\\ H_i(K;\mathbf{Z}/2) &= (0\otimes\mathbf{Z}/2)\oplus\mathrm{Tor}(0,\mathbf{Z}/2) = 0 \quad \text{if } i > 2, \end{aligned}$$

which agrees with our earlier calculation.

The universal coefficient theorem shows that integral homology contains all the information that $\mathbf{Z}/2$ homology contains, while the Klein bottle shows that it actually contains *more* information, as only integral homology can distinguish the Klein bottle from the torus.

EXERCISES

9.1. Calculate the mod 2 homology of the simplicial annulus of Example 7.3.

9.2. Triangulate the closed interval $[0,1]$ and calculate its mod 2 homology. Verify that the result does not change if you use a different triangulation.

9.3. Triangulate the cylinder $S^1 \times [0,1]$ and calculate its mod 2 homology and its integral homology.

9.4. Triangulate the Möbius band and calculate its mod 2 homology and its integral homology. Verify that the universal coefficient theorem holds for this space.

9.5. Calculate the integral homology of the simplicial annulus, of the simplicial square of Example 7.2, and of the simplicial sphere of Example 7.11, and verify the results given in Examples 9.12, 9.13 and 9.14.

9.6. Take the simplicial torus of Example 7.8, and glue in another 2-simplex joining the three innermost edges. How does gluing in this 2-simplex change the homology? Compare the change in the homology with the change in the Euler characteristic.