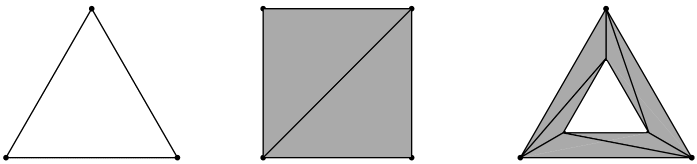
7
The Euler Number

We now begin our study of topological invariants, by considering the "Euler number" or "Euler characteristic." This assigns an integer to each topological space in a way that tells us something about the topology of the space. In particular, it can sometimes tell if two spaces are not homotopy equivalent, since spaces which are homotopy equivalent have the same Euler number.

7.1 Simplicial Complexes

Although it is possible to define the Euler number for all spaces, for clarity we will begin by restricting our attention to "simplicial complexes." These are spaces built out of cells, called simplices.[1] For example, here are simplicial complexes homeomorphic to the circle, the solid square, and the annulus.



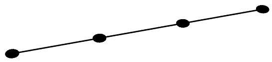
These have been built out of points, lines and triangles, i.e., 0-simplices, 1-simplices and 2-simplices. Essentially, a k-simplex is described by a list of $k+1$

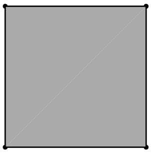
[1] For reasons which I cannot fathom, the plural of "simplex" is "simplices" whereas the plural of "complex" is "complexes"!

M.D. Crossley, *Essential Topology*, Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84628-194-5_7,

vertices (points in some $\mathbf{R}^n$), and is the smallest convex subspace of $\mathbf{R}^n$ containing those vertices. Hence, for example, a 1-simplex is defined by 2 vertices and is the straight line from one of these vertices to the other.

However, we need to be careful about our choice of vertices in order to avoid degenerate situations such as the following:





These both have 4 vertices, so would be 3-simplices by the above definition, whereas neither is 3-dimensional.

To avoid such cases, we insist that the vertices $v_0, \ldots, v_k$ be in **general position**, by which we mean that the k vectors

$$v_1 - v_0, \quad v_2 - v_1, \quad \ldots, \quad v_k - v_{k-1}$$

in $\mathbf{R}^n$ are linearly independent. This ensures that a k-simplex really is k-dimensional, i.e., is not contained in any $(k-1)$-dimensional subspace of $\mathbf{R}^n$.

Thus we define a k**-simplex** as the smallest convex subspace of $\mathbf{R}^n$ containing a given list of $k+1$ vertices which are in general position. We will write $[v_0, \ldots, v_k]$ for the k-simplex with vertices $v_0, \ldots, v_k$, so that $[v_0, \ldots, v_k]$ consists of all linear combinations

$$t_0 v_0 + \cdots + t_k v_k$$

where the coefficients $t_0, \ldots, t_k$ are real numbers between 0 and 1 satisfying $t_0 + \cdots + t_k = 1$ and are called the **barycentric coordinates** of the point $t_0 v_0 + \cdots + t_k v_k$.

If we have a k-simplex with vertices $v_0, \ldots, v_k$, then any non-empty subset of these vertices will also determine a simplex, called a **subsimplex** of the original k-simplex. We will say that a subsimplex is a **face**[2] if it only omits one vertex, so a k-simplex will have $k+1$ faces and $2^{k+1}-1$ subsimplices. For example, a 2-simplex $[v_0, v_1, v_2]$ has 7 subsimplices: $[v_0]$, $[v_1]$, $[v_2]$, $[v_0, v_1]$, $[v_0, v_2]$, $[v_1, v_2]$ and $[v_0, v_1, v_2]$, of which 3 are faces: $[v_0, v_1]$, $[v_0, v_2]$, $[v_1, v_2]$. The union of the faces is called the **boundary** of the simplex. The complement of the boundary is called the **interior** of the simplex. The boundary consists of all points with at least one barycentric coordinate equal to 0, and the interior consists of all points whose barycentric coordinates are *all* non-zero. For 0-simplices (i.e., points), the boundary is actually empty, since a 0-simplex has no proper subsimplices, so the interior is the simplex itself in this case.

[2] Some authors use the term "face" to mean subsimplex, but we reserve the word "face" for a $(k-1)$-subsimplex of a k-simplex

A simplicial complex is, essentially, just a finite union of simplices. However, to avoid some technical problems later, we will insist on two extra conditions.

Definition: A **simplicial complex** K is a subspace of $\mathbf{R}^n$ together with a finite list of simplices such that:
1. The union of the simplices is the set K and each point in K lies in the interior of only one simplex.
2. Every face of every simplex in the list is also in the list.

Note that some books allow a simplicial complex to have infinitely many simplices. For most examples, a finite number of simplices is enough and ensures that every simplicial complex is compact, being a finite union of compact sets.

We say that a simplicial complex is n-dimensional if it has at least one n-simplex, but no $(n+1)$-simplices, $(n+2)$-simplices, etc.

Example 7.1

The simplicial circle above is a one-dimensional simplicial complex with three 0-simplices (the vertices of the triangle) and three 1-simplices (the edges of the triangle).

It is tempting to think that one can make a simplicial circle with only two vertices and two curved edges. However, since simplices must be convex, we cannot have curved 1-simplices. Hence we must have at least three vertices in a simplicial circle.

Example 7.2

The simplicial square above is a two-dimensional simplicial complex with four 0-simplices, five 1-simplices and two 2-simplices.

Example 7.3

The simplicial annulus above is a two-dimensional simplicial complex with 6 0-simplices, 12 1-simplices and 6 2-simplices.

With a little work, we can see that the conditions in our definition of simplicial complex ensure that non-empty intersections of simplices are always simplices.

Proposition 7.4

If S and T are simplices of a simplicial complex K, then $S \cap T$ is either empty or a subsimplex of both S and T.

Proof

If $S \cap T$ is not empty, let $v_1, \ldots, v_n$ be the set of all vertices of K that are contained in $S \cap T$. We will prove that $S \cap T$ is a subsimplex of both S and T by showing that $S \cap T$ is the simplex $[v_1, \ldots, v_n]$.

To do this, let $x \in S \cap T$ be any point. By condition 1 of the definition of simplicial complex, x is contained in the interior of exactly one simplex of K. Let $[w_1, \ldots, w_k]$ be that simplex. Since $x \in S$ we can write x as a linear combination of the vertices of S, with non-negative real coefficients that sum to 1. By taking only those vertices of S that have non-zero coefficients in this expression, we can find a subsimplex of S whose interior contains x. A subsimplex of S is a simplex of K by applying condition 2 repeatedly, so this subsimplex must be $[w_1, \ldots, w_k]$ by the uniqueness in condition 1. Hence $w_1, \ldots, w_k$ are vertices in S. The same argument can be applied with T in place of S, from which we see that $w_1, \ldots, w_k \in S \cap T$. Thus $\{w_1, \ldots, w_k\} \subset \{v_1, \ldots, v_n\}$, so $x \in [w_1, \ldots, w_k] \subset [v_1, \ldots, v_n]$. Consequently $S \cap T \subset [v_1, \ldots, v_n]$. On the other hand, since $v_1, \ldots, v_n \in S$, and S is convex, we have $[v_1, \ldots, v_n] \subset S$. The same argument applies to T, revealing that $[v_1, \ldots, v_n] \subset S \cap T$. Hence these two sets coincide. □

7.2 The Euler Number

If we have a simplicial complex, we associate a number to it, the "Euler number" in the following way.

Definition: If T is an n-dimensional simplicial complex and, for each k, i_k is the number of k-simplices in T, then the **Euler number** of T, written $\chi(T)$, is given by

$$\chi(T) = i_0 - i_1 + i_2 - i_3 + \cdots + (-1)^n i_n.$$

In many books the Euler number is called the **Euler characteristic**; we will use the two terms interchangeably.

Example 7.5

The simplicial circle with three 0-cells and three 1-cells has Euler number $\chi = 3 - 3 = 0$.

Example 7.6

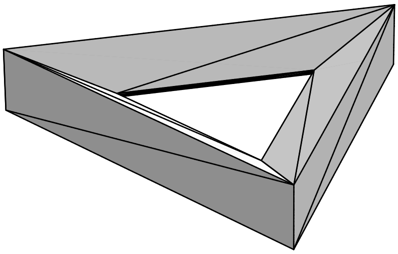
The simplicial square with four 0-cells, five 1-cells and two 2-cells has Euler number $\chi = 4 - 5 + 2 = 1$.

Example 7.7

The simplicial annulus with 6 0-cells, 12 1-cells and 6 2-cells has Euler number $\chi = 6 - 12 + 6 = 0$.

Example 7.8

There is a simplicial torus which looks like this:

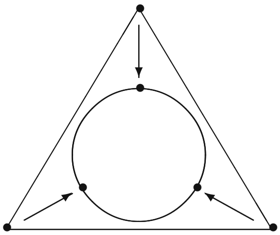


This has 9 0-simplices, 27 1-simplices and 18 2-simplices, so its Euler number is $\chi = 9 - 27 + 18 = 0$.

Having defined the Euler characteristic of a simplicial complex, we can extend this to topological spaces by an appropriate homeomorphism, called a "triangulation". A **triangulation** of a topological space T is a simplicial complex K and a homeomorphism $K \leftrightarrow T$. A space for which such a triangulation exists is said to be **triangulable**.

Example 7.9

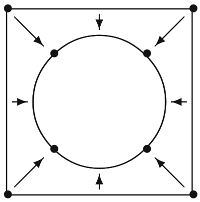
One triangulation of the circle S^1 is given by the simplicial complex in $\mathbf{R}^2$ which has three 0-simplices $(0, 2)$, $(\sqrt{3}, -1)$, $(-\sqrt{3}, -1)$ and three 1-simplices between them, together with a homeomorphism between this and S^1, such as $(x, y) \longrightarrow (x, y)/\sqrt{x^2 + y^2}$.



This simplicial complex has Euler number $3 - 3 = 0$.

Example 7.10

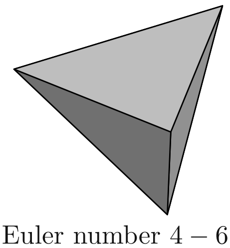
Another triangulation of S^1 can be given by taking the simplicial complex K in $\mathbf{R}^2$ that has four 0-simplices $(2, 0)$, $(-2, 0)$, $(0, 2)$, $(0, -2)$ and four 1-simplices between them, with a homeomorphism, such as $(x, y) \mapsto (x, y)/\sqrt{x^2 + y^2}$.



This simplicial complex has Euler number $4 - 4 = 0$.

Example 7.11

We can triangulate the 2-sphere S^2 as a tetrahedron, using four 2-simplices, six 1-simplices and four 0-simplices.



This simplicial complex has Euler number $4 - 6 + 4 = 2$.

Notice that the two different triangulations of S^1 both have the same Euler number. Indeed, it is easy to convince yourself that *any* triangulation of S^1 will have Euler number 0. In fact, the analogous result is true for any triangulable space: Any two different triangulations of the same space will have the same Euler number, although this is very difficult to prove. (One way to prove it is to use Lemma 6.11 and Theorem 7.13 below.) This allows us to define the

Euler number $\chi(T)$ of a triangulable space T to be the Euler number of any simplicial complex K homeomorphic with T. We can then deduce that if two spaces have different Euler numbers, then they cannot be homeomorphic.

Example 7.12

The sphere S^2 has Euler number 2, hence S^2 is not homeomorphic with the torus T^2 which has Euler number 0.

Moreover, we can deduce that two triangulable spaces with different Euler numbers cannot even be homotopy equivalent, thanks to the following result.

Theorem 7.13

If two triangulable spaces are homotopy equivalent, then they have the same Euler number.

This is also very difficult to prove, of course, and we will derive it as a consequence of another very deep theorem at the end of Chapter 10.

Example 7.14

The sphere S^2 has Euler number 2, hence S^2 is not homotopy equivalent with the torus T^2 which has Euler number 0. Moreover, neither S^2 nor T^2 is contractible, since a one-point space has Euler number 1.

7.3 The Euler Characteristic and Surfaces

We have seen that two homeomorphic spaces have the same Euler number. However, many pairs of spaces which are not homeomorphic, or even homotopy equivalent, have the same Euler number. For example, $\chi(S^1) = 0 = \chi(T^2)$, yet the circle and the torus are not homotopy equivalent, as we will see in Example 8.11.

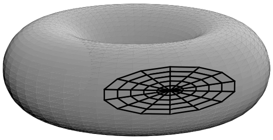
However, for certain types of spaces, the Euler number can distinguish non-homeomorphic spaces. The most important example of this is the classification of triangulable "surfaces." A **surface** is defined to be a Hausdorff space with the property that around every point in the space, there is an open neighbourhood homeomorphic with an open disc in $\mathbf{R}^2$.

Example 7.15

The plane $\mathbf{R}^2$ is a surface, since every point (x, y) is contained in a neighbourhood, say $B_1(x, y)$, which is an open disc in $\mathbf{R}^2$.

Example 7.16

The torus T^2 is a surface. For example, the picture below depicts a disc-like neighbourhood about the point $(3/\sqrt{2}, -3/\sqrt{2}, 0)$.



Example 7.17

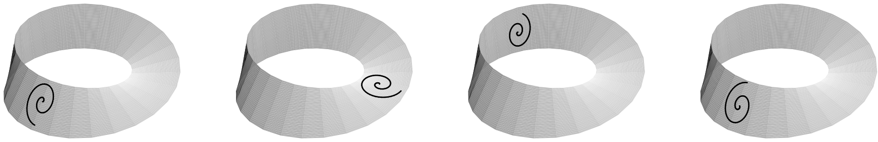
Similarly, the sphere S^2 is a surface, as are the genus 2 surface of Example 3.27 and the Klein bottle of Example 5.57.

Example 7.18

The open cylinder $S^1 \times (0, 1)$ is a surface similarly. And if we form a Möbius band out of this by cutting, twisting and gluing, then we will get another surface.

Now, a surface may be orientable or non-orientable. To understand the difference, imagine holding a small sheet of paper against the surface at one point, with an asymmetric figure, such as a spiral, drawn on the paper.

We can slide the paper along the surface and rotate it around and, provided that we do not move very far, the spiral will look the same. However, on some surfaces it is possible to slide the paper around the surface in a certain way and arrive back at the starting point with the spiral reflected. For example, on a Möbius band, if you slide the paper once around the band, this will happen.



In fact, if you do this with a real Möbius band and piece of paper, the paper will end up on the opposite side of the band to where you first started, but we want to imagine this sliding process taking place *inside* the surface, so we will suppose that the surface and the paper are transparent so that we cannot tell which side of the surface the paper is on.

We then say that the Möbius band is **non-orientable** because the orientation of the spiral does not stay constant as you slide around the surface. By contrast, a cylinder is **orientable** as, no matter how you move around the surface, the spiral will have the same orientation.

With this notion of orientability, we have the following amazing result.

Theorem 7.19 (Classification of Surfaces)

Two triangulable surfaces S and T are homeomorphic if, and only if, they have the same Euler number and the same orientability (i.e., either both are orientable or both are non-orientable).

In other words, if we know whether the surfaces are orientable or not, then their Euler number is enough to tell whether the surfaces are homeomorphic to each other or not.

As we have defined things, the condition that S and T be triangulable is necessary for their Euler numbers to be defined. However, it also hides an important hypothesis, which is that S and T be compact for, as we mentioned earlier, every simplicial complex is compact. The theorem, consequently, does not apply to the spaces of Examples 7.15 and 7.18. Some people use the term **closed surface** to mean a compact surface, and so will talk of this theorem as giving a classification of closed surfaces.

The proof of Theorem 7.19 is too long for this book, but it is explained very well in Chapter 1 of [7] and Chapter 17 of [4].

EXERCISES

7.1. Show that every triangulable space is Hausdorff.

7.2. Show directly that any two triangulations of the circle S^1 have the same Euler number. Which other spaces can you give such a direct proof for?

7.3. Using any triangulation that you can think of, calculate the Euler number of (1) a closed interval $[a, b]$, (2) a cylinder, (3) a Möbius band, (4) a surface of genus two, (5) the Klein bottle.

7.4. For each positive integer n, find a simplicial complex with Euler number n. For each positive integer n, try to find a connected simplicial complex with Euler number n.

7.5. Which integers (positive or negative) can occur as the Euler number of a one-dimensional simplicial complex? Which integers can occur as the Euler number of a connected one-dimensional simplicial complex?

7.6. Give an example of a 'non-Hausdorff surface', i.e., a topological space S which is not Hausdorff, but which has the property that every point has an open neighbourhood homeomorphic with an open disc in $\mathbf{R}^2$.