from $\mathbf{H}$ *to* $\mathcal{U}$

Minh-Tâm Quang Trinh

Massachusetts Institute of Technology

G split semisimple group

W Weyl group

Br_W braid group

V reflection representation of W

Motivations for this talk:

- Invariants of braids and knots/links
- Topology of varieties related to G
- Representations of algebras related to $W \curvearrowright V$

$\mathbf{A}_W = \mathbf{C}[W] \ltimes \mathrm{Sym}(V)$

We'll study a monoidal trace

$$\mathrm{AH} : \mathbf{H}_W^{op} \to \mathbf{Mod}_2(\mathbf{A}_W)$$

and its decategorification

$$[-]_q = \varepsilon \cdot \sum_{i,j} (-1)^i q^{i+j} (\mathrm{AH}^{i,j})^\vee : H_W \to R_W[\![q]\!],$$

where:

- $\mathbf{H}_W$ is the Hecke category of W
- $\mathbf{Mod}_2(\mathbf{A}_W)$ is a category of bigraded $\mathbf{A}_W$-modules
- H_W is the Hecke algebra of W
- R_W is the representation ring of W

Thm 1 Khovanov–Rozansky's HHH factors as

$$\mathbf{H}_W \xrightarrow{\mathrm{AH}^\vee} \mathbf{Mod}_2(\mathbf{A}_W) \xrightarrow{\mathrm{Hom}_W(\Lambda^*(V), -)} \mathbf{Vect}_3.$$

Thus, AH is related to link invariants when $W = S_n$.

$\widetilde{\mathcal{U}} \to \mathcal{U}$ the Springer resolution

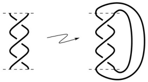
Thm 2 For a positive braid $\beta \in Br_W^+$,

$$\mathrm{AH}^{i,j}(\mathcal{R}(\beta)) = \mathrm{gr}_j^{\mathbf{W}}\, \mathrm{H}_i^{!,G}(\mathcal{Z}(\beta)),$$

where:

- $\mathcal{R}(\beta) \in \mathbf{H}_W$ is the Rouquier complex of β
- $\mathcal{Z}(\beta) = \widetilde{\mathcal{U}} \times_{\mathcal{U}} \mathcal{U}(\beta)$ is a generalized Steinberg variety

Example A topological braid β has a *link closure* $\hat{\beta}$:



A link has a *superpolynomial* $\mathbf{P} \in \mathbf{Z}(q^{\pm 1/2})[a^{\pm 1}, t^{\pm 1}]$.

If $\beta = (\sigma_1\sigma_2\sigma_3)^3 \in Br_4$, then $\hat{\beta}$ is the $(3,4)$ torus knot and

$$\begin{aligned}\mathbf{P}(\hat{\beta}) = a^6 q^{-3}&(1 + q^2t^2 + q^3t^4 + q^4t^4 + q^6t^6)\\ &+ a^8q^{-2}(t^3 + qt^5 + q^2t^5 + q^3t^7 + q^4t^8)\\ &+ a^{10}t^8.\end{aligned}$$

Thms 1, 2 imply that up to a shift, the red term is

$$\sum_{i,j} q^j t^i\, \mathrm{gr}_j^{\mathbf{W}}\, \mathrm{H}_i^{!,G}(\mathcal{U}(\beta)).$$

Thm 3 For positive β,

$$[\beta]_q = \pm \frac{1}{|G(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{U}(\beta)_u(\mathbf{F}_q)| [\mathcal{B}_u]_q,$$

where $\mathcal{B}_u$ is the Springer fiber over u and

$$[\mathcal{B}_u]_q = \sum_i q^i \mathrm{H}^{2i}(\mathcal{B}_u),$$

an element of $R_W[q]$.

$\mathbf{D}_\nu^{\mathrm{rat}} \supseteq \mathbf{A}_W$ rational DAHA of slope $\nu \in \mathbf{Q}$

Thm 4 For *periodic* β of "good" slope $\nu > 0$, $[\beta]_q$ is the q-character of an explicit $\mathbf{D}_\nu^{\mathrm{rat}}$-module.

Example Write $\mathrm{Irr}(S_4) = \{1, \phi, \psi, \varepsilon\phi, \varepsilon\}$ with $\phi = \mathrm{tr}(- \mid V)$.

If $\beta = (\sigma_1\sigma_2\sigma_3)^3 \in Br_4$, as before, then

$$\begin{aligned} [\beta]_q = {} & (1 + q^2 + q^3 + q^4 + q^6) \cdot 1 \\ & + (q + q^2 + q^3 + q^4 + q^5) \cdot \phi \\ & + (q^2 + q^4) \cdot \psi \\ & + q^3 \cdot \varepsilon\psi. \end{aligned}$$

Thm 3 claims this is a sum of $[\mathcal{B}_u]_q$. Indeed:

$$[\beta]_q = q^6 \cdot [\mathcal{B}_4]_q + (q^3 + q^4) \cdot [\mathcal{B}_{3,1}]_q + q^2 \cdot [\mathcal{B}_{2,2}]_q + 1 \cdot [\mathcal{B}_{2,1,1}]_q + 0 \cdot [\mathcal{B}_{1,1,1,1}]_q.$$

Thm 4 claims this is also the q-character of a certain $\mathbf{D}_{3/4}^{\mathrm{rat}}$-module. It's the simple quotient of $\mathrm{Sym}(V)$.

Thm 1 was inspired by Webster–Williamson's geometric model for Khovanov–Rozansky homology.

We expect AH to match the (underived) horizontal trace on $\mathbf{H}_W$ studied by Gorsky–Hogancamp–Wedrich and others.

Thms 2, 3 came from asking how Springer theory interacts with nonabelian Hodge phenomena, which certain stacks $\mathcal{U}(\beta)/G$ and $\mathcal{Z}(\beta)/G$ should exhibit.

Inspired by Yun, Oblomkov–Rasmussen–Shende, Shende–Treumann–Zaslow…

Thm 4 came from Gorsky–Oblomkov–Rasmussen–Shende's conjectures relating $\mathbf{D}_\nu^{\mathrm{rat}}$-modules and KR of torus knots.

To describe AH, we interpret the Hecke category geometrically. Henceforth, subscript 0 means "over $\mathbf{F}_q$." No subscript means "over $\bar{\mathbf{F}}_q$."

$\mathcal{B}_0$ flag variety of (the split form) G_0

By a theorem of Iwahori, the Hecke algebra is

$$\begin{aligned} H_W \otimes \mathbf{C}(q^{1/2}) &\simeq \mathrm{End}_{G(\mathbf{F}_q)}(\mathbf{C}[\mathcal{B}(\mathbf{F}_q)]) \\ &\simeq \mathbf{C}[\mathcal{B}(\mathbf{F}_q) \times \mathcal{B}(\mathbf{F}_q)]^{G(\mathbf{F}_q)}. \end{aligned}$$

Similarly, the Hecke category is built from

$$\mathrm{D}^b_{G,m}(\mathcal{B}_0 \times \mathcal{B}_0),$$

the bounded derived category of G_0-equivariant mixed complexes of sheaves over $\mathcal{B}_0 \times \mathcal{B}_0$ with constructible cohomology.

The G-orbits of $\mathcal{B} \times \mathcal{B}$ are indexed by W:

$$\mathcal{B}_0 \times \mathcal{B}_0 = \coprod_{w \in W} O_{w,0}.$$

Each $O_{w,0}$ defines an intersection complex $IC_{w,0}$.

The Hecke category is $\mathbf{H}_W = \mathrm{K}^b(\mathrm{C}(\mathcal{B}_0 \times \mathcal{B}_0))$, where

$$\mathrm{C}(\mathcal{B}_0 \times \mathcal{B}_0) = \left\langle IC_{w,0}\langle n \rangle : \begin{array}{l} w \in W, \\ n \in \mathbf{Z} \end{array} \right\rangle_{\oplus} \subseteq \mathrm{D}^b_{G,m}(\mathcal{B}_0 \times \mathcal{B}_0).$$

There is a geometric convolution on $\mathrm{C}(\mathcal{B}_0 \times \mathcal{B}_0)$.

The $IC_{w,0}$ decategorify to the Kazhdan–Lusztig basis.

The *shift-twist* $\langle 1 \rangle = [1](\frac{1}{2})$ decategorifies to $q^{-1/2}$.

Lusztig introduced the diagram below to study unipotent representations of G:

$$\mathcal{B}_0 \times \mathcal{B}_0 \xleftarrow{act} G_0 \times \mathcal{B}_0 \xrightarrow{pr} G_0$$

The functor

$$\mathrm{CH} = \bigoplus_i {}^p\mathcal{H}^i[-i] \circ pr_! \circ act^* : \mathrm{D}^b_{G,m}(\mathcal{B}_0^2) \to \mathrm{D}^b_{G,m}(G_0)$$

descends to a monoidal trace on $\mathbf{H}_W$.

Webster–Williamson showed that

$$\mathrm{gr}^{\mathbf{W}}_{i+j} \mathrm{H}^i_G(G, \mathrm{CH}(IC_w)) \simeq \mathrm{HH}^j(\mathrm{H}^{i+j}_G(\mathcal{B} \times \mathcal{B}, IC_w)).$$

HH^* is Hochschild homology over $\mathrm{H}^*_G(\mathcal{B}) \simeq \mathrm{Sym}(V)$.

So, Khovanov–Rozansky's HHH factors as

$$\mathbf{H}_W = \mathrm{K}^b(\mathrm{C}(\mathcal{B}_0 \times \mathcal{B}_0)) \xrightarrow{\mathrm{CH}} \mathrm{K}^b(\mathrm{C}(G_0)) \xrightarrow{\mathrm{gr}_*^{\mathbf{W}}\, \mathrm{H}_G^*} \mathbf{Vect}_3$$

where

$$\mathrm{C}(G_0) = \left\langle E_0 : \begin{array}{l} E_0 \text{ is a subquotient} \\ \text{of } \mathrm{CH}(IC_{w,0})\langle n\rangle \\ \text{for some } w, n \end{array} \right\rangle_{\oplus} \subseteq \mathrm{D}^b_{G,m}(G_0).$$

But the objects of the $\mathrm{K}^b(\mathrm{C}(-))$ are not directly related to the topology of actual varieties, in general.

We need a *realization functor*

$$\rho : \mathrm{K}^b(\mathrm{C}(-)) \to \mathrm{D}^b_{G,m}(-)$$

to relate them to actual geometric objects.

A sufficient condition for ρ to exist is:

$$i \text{ nonzero} \implies \mathrm{gr}_0^{\mathbf{W}} \operatorname{Ext}^i(K, L) = 0$$

for all $K_0, L_0 \in \mathrm{C}(-)$.

This fails for $\mathrm{C}(G_0)$. But by work of Lusztig and Rider–Russell, it holds for

$$\mathrm{C}(\mathcal{U}_0) = \langle \iota^* E_0 : E_0 \in \mathrm{C}(G_0)\rangle_{\oplus} \subseteq \mathrm{D}^b_{G,m}(\mathcal{U}_0),$$

where $\iota : \mathcal{U}_0 \to G_0$ is the inclusion. Thus a diagram:

$$\begin{array}{ccccc} \mathrm{K}^b(\mathrm{C}(\mathcal{B}_0^2)) & \xrightarrow{\mathrm{CH}} & \mathrm{K}^b(\mathrm{C}(G_0)) & \xrightarrow{\iota^*} & \mathrm{K}^b(\mathrm{C}(\mathcal{U}_0)) \\ \downarrow \rho & & & & \downarrow \rho \\ \mathrm{D}^b_{G,m}(\mathcal{B}_0^2) & \xrightarrow{\mathrm{CH}} & \mathrm{D}^b_{G,m}(G_0) & \xrightarrow{\iota^*} & \mathrm{D}^b_{G,m}(\mathcal{U}_0) \end{array}$$

There are special objects in $\mathrm{C}(G_0)$ and $\mathrm{C}(\mathcal{U}_0)$:

- $\mathcal{G}_0 \in \mathrm{C}(G_0)$, the Grothendieck–Springer sheaf
- $\mathcal{S}_0 \in \mathrm{C}(\mathcal{U}_0)$, the Springer sheaf

By a theorem of Lusztig, $\mathbf{A}_W \simeq \mathrm{Ext}^*(\mathcal{S}, \mathcal{S})$.

Our functor AH is the composition

$$\mathbf{H}_W^{op} = \mathrm{K}^b(\mathrm{C}(\mathcal{B}_0^2))^{op} \to \mathrm{D}^b_{G,m}(\mathcal{U}_0)^{op} \xrightarrow{\mathrm{gr}_*^{\mathbf{W}} \mathrm{Ext}^*(-,\mathcal{S})} \mathbf{Mod}_2(\mathbf{A}_W).$$

Use the top half of the diagram to show **Thm 1**:

$$\mathrm{Hom}_W(\Lambda^*(V), \mathrm{AH}^\vee) \simeq \mathrm{HHH}.$$

Key step is $\bigoplus_i \mathrm{gr}_i^{\mathbf{W}} \mathrm{Ext}^i(\mathcal{G}, \mathcal{G}) \simeq \mathrm{Ext}^*(\mathcal{S}, \mathcal{S})$.

Thm 2: For positive β,

$$\mathrm{AH}(\mathcal{R}(\beta)) \simeq \mathrm{gr}_*^{\mathbf{W}} \mathrm{H}_*^{!,G}(\mathcal{Z}(\beta)).$$

We need to define $\mathcal{R}(\beta)$ and $\mathcal{Z}(\beta)$.

Broué–Michel and Deligne introduced a map

$$O : Br_W^+ \to \left\{ \begin{array}{l} G_0\text{-varieties over } \mathcal{B}_0 \times \mathcal{B}_0 \\ \text{up to strict isomorphism} \end{array} \right\}$$

such that $O(\alpha\beta)_0 \simeq O(\alpha)_0 \times_{\mathcal{B}_0} O(\beta)_0$.

The complex $\mathcal{R}(\beta) \in \mathbf{H}_W$ is characterized by

$$\rho(\mathcal{R}(\beta)) = (O(\beta)_0 \to \mathcal{B}_0 \times \mathcal{B}_0)_! \mathbf{C}.$$

The variety $\mathcal{Z}(\beta)_0$ is a certain pullback of $O(\beta)_0$.

We define $\mathcal{U}(\beta)_0$ and $\mathcal{Z}(\beta)_0$ by cartesian squares:

$$\begin{array}{ccccc} O(\beta)_0 & \longleftarrow & \mathcal{U}(\beta)_0 & \longleftarrow & \mathcal{Z}(\beta)_0 \\ {\scriptstyle j_\beta}\downarrow & & \downarrow & & \downarrow \\ \mathcal{B}_0 \times \mathcal{B}_0 & \xleftarrow{act} & \mathcal{U}_0 \times \mathcal{B}_0 & \longleftarrow & \widetilde{\mathcal{U}_0} \times \mathcal{B}_0 \end{array}$$

In particular, $\widetilde{\mathcal{U}_0} = \mathcal{U}(\mathbf{1})_0$ for the identity braid $\mathbf{1}$.

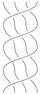
$\mathbf{A}_W \curvearrowright \mathrm{AH}(\mathcal{R}(\beta))$ from $\mathrm{H}^{!,G}_*(\mathcal{Z}(\mathbf{1})) \curvearrowright \mathrm{H}^{!,G}_*(\mathcal{Z}(\beta))$.

Cor Up to (pure) shifts,

- The bottom a-degree of HHH matches $\mathrm{gr}^{\mathbf{W}}_* \mathrm{H}^{!,G}_*(\mathcal{U}(\beta))$.
- The top a-degree of HHH matches $\mathrm{gr}^{\mathbf{W}}_* \mathrm{H}^{!,G}_*(\mathcal{X}(\beta))$, where

$$\mathcal{X}(\beta)_0 = \mathcal{U}(\beta)_0 \times_{\mathcal{U}_0} \{1\}.$$

The *full twist* is a central element $\pi = \sigma^2_{w_0} \in Br^+_W$:



Gorsky–Hogancamp–Mellit–Nakagane proved

bottom a-degree of $\mathrm{HHH}(\beta) \simeq$ top a-degree of $\mathrm{HHH}(\beta\pi)$,

refining a theorem of Kálmán.

Cor For positive β,

$$\mathrm{gr}^{\mathbf{W}}_* \mathrm{H}^{!,G}_*(\mathcal{U}(\beta)) \simeq \mathrm{gr}^{\mathbf{W}}_* \mathrm{H}^{!,G}_*(\mathcal{X}(\beta\pi)).$$

What is a geometric explanation for this isomorphism?

It's *not* induced by a homemorphism of stacks, in general.

Thm 3 is a decategorified analogue of **Thm 2**:

$$[\beta]_q = \pm \frac{1}{|G(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{U}(\beta)_u(\mathbf{F}_q)| [\mathcal{B}_u]_q,$$

where $[\mathcal{B}_u]_q = \sum_i q^i \mathrm{H}^{2i}(\mathcal{B}_u)$. However, it is *not* just a corollary.

The virtual weight series of $[X/G]$ need *not* be the quotient of the virtual weight polynomial of X by that of G.

Instead, the proof uses a strange formula

$$[\beta]_q = q^{|\beta|/2} \varepsilon \cdot \sum_i q^i \operatorname{Sym}^i(V) \cdot \sum_{\phi, \psi \in \mathrm{Irr}(W)} \{\phi, \psi\} \phi_q(\beta) \psi,$$

where $\{-,-\} : \mathrm{Irr}(W)^2 \to \mathbf{Q}$ is Lusztig's "exotic Fourier transform."

Cor For parabolic $W' \subseteq W$, we have a commutative diagram

$$\begin{array}{ccc} H_{W'} & \xrightarrow{[-]_q} & R_{W'}[\![q]\!] \\ \downarrow & & \downarrow (1-q)^{-d} \mathrm{Ind}_{W'}^{W} \\ H_W & \xrightarrow{[-]_q} & R_W[\![q]\!] \end{array}$$

where $d = \mathrm{rk}(W) - \mathrm{rk}(W')$.

This is a kind of Markov property for $[-]_q$. The proof uses an induction formula for the $[\mathcal{B}_u]_q$.

Cor If $\beta \mapsto w$ under $Br_W^+ \to W$, then

$$[\beta]_q \in \frac{1}{(1-q)^{\dim(V^w)}} R_W[q].$$

The proof uses a result of Lusztig on the sizes of G-stabilizers.

Example Writing $r = \mathrm{rk}(W)$, we compute

$$[\mathbf{1}]_q = \frac{1}{(1-q)^r} \operatorname{Ind}_{\{1\}}^W([\mathbf{1}]_q) = \frac{1}{(1-q)^r} \mathbf{C}[W].$$

For $W = S_n$, recovers:

$$\mathbf{P}(n\text{-unlink})_{t=-1} = \left(\frac{a - a^{-1}}{q^{1/2} - q^{-1/2}}\right)^{n-1}.$$

Example Let
$W = S_3 = \langle s, t : s^2 = t^2 = (sts)^2 = 1\rangle$.

Writing $\mathrm{Irr}(S_3) = \{1, \phi, \varepsilon\}$, we compute

$$[\sigma_w]_q = \begin{cases} (1-q)^{-2}(1 + 2\phi + \varepsilon) & w = 1 \\ (1-q)^{-1}(1+\phi) & w \in \{s,t\} \\ 1 & w \in \{st, ts\} \\ (1-q)^{-1}(1 - q + q^2 + q\phi) & w = sts \end{cases}$$

A braid β is *periodic of slope* $\frac{m}{n} \in \mathbf{Q}$ iff: $\beta^n = \pi^m$.

Using the "exotic" formula, we can show:

Lem If β is periodic of slope $\nu \in \mathbf{Q}$, then

$$[\beta]_q = \sum_{\phi \in \mathrm{Irr}(W)} q^{\nu \mathbf{c}(\phi)} \mathrm{Deg}_\phi(e^{2\pi i \nu})\phi \cdot \sum_i q^i \operatorname{Sym}^i(V),$$

where:

- $\mathrm{Deg}_\phi(q) \in \mathbf{Q}[q]$ is the degree of the *unipotent principal series* of $G(\mathbf{F}_q)$ attached to ϕ.
- $\mathbf{c}(\phi)$ is the *content* of ϕ. For $W = S_n$, it's the content of the corresponding partition.

The key is that the traces $\phi_q(\beta)$ are computable.

This goes back to Jones's formula for HOMFLY of torus knots.

The *rational DAHA* is a deformation of $\mathbf{C}[W] \ltimes \mathcal{D}(V)$, where $\mathcal{D}(V)$ is the Weyl algebra of V:

$$\mathbf{D}_\nu^{\mathrm{rat}} = \frac{\mathbf{C}[W] \ltimes (\mathbf{C}[V] \otimes \mathbf{C}[V^\vee])}{[x,y] - \langle x,y\rangle - \nu \sum_{\alpha\in\Phi^+} \langle x,\alpha^\vee\rangle\langle \alpha,y\rangle s_\alpha}.$$

It enjoys a well-behaved "category O" of modules where:

- Simple modules $L_\nu(\phi)$ are indexed by $\phi \in \mathrm{Irr}(W)$.
- Each module M admits a W-stable grading, giving us $[M]_q \in R_W(q^{1/2})$.

There is a *Knizhnik–Zamolodchikov functor*

$$\mathbf{Mod}(\mathbf{D}_\nu^{\mathrm{rat}}) \to \mathbf{Mod}(H_W|_{q^{1/2}=e^{\pi i\nu}}),$$

hinting that the lemma is related to $\mathbf{D}_\nu^{\mathrm{rat}}$.

Each simple $L_\nu(\phi)$ is the quotient of a Verma $\Delta_\nu(\phi)$.

If β is periodic of slope ν, then

$$[\beta]_q = (q^{1/2})^{\nu|\Phi|-r} \cdot \sum_{\phi\in\mathrm{Irr}(W)} \mathrm{Deg}(e^{2\pi i\nu})[\Delta_\nu(\phi)]_q.$$

Let n be the denominator of $\nu \in \mathbf{Q}$ in lowest terms.

- If n is *elliptic*, then $[\beta]_q \in R_W[q]$ by our earlier result. This implies Varagnolo–Vasserot's result

$$n \text{ elliptic} \implies \dim L_\nu(1) < \infty.$$

- **Thm 4.** For W irreducible and n *cuspidal*,

$$[\beta]_q = \begin{cases} [L_\nu(1)] + [L_\nu(V)] & (W,n) = (E_8,15),(H_4,15) \\ [L_\nu(1)] & \text{else} \end{cases}$$

  The proof uses the KZ functor and the block theory of H_W.

Does this character come from $\mathbf{D}_\nu^{\mathrm{rat}} \curvearrowright \mathrm{H}_*^{!,G}(\mathcal{Z}(\beta))$?

More complicated:

- $\mathbf{A}_W = \mathbf{A}_{W,0}$, where $\mathbf{A}_{W,\varpi} = \mathrm{H}_*^{!,G\times\mathbf{G}_m}(\mathcal{Z}(\mathbf{1}))$.
- $\mathbf{D}_\nu^{\mathrm{rat}} = \mathbf{D}_{\nu,1}^{\mathrm{rat}}$ for some $\mathbf{D}_{\nu,\varpi}^{\mathrm{rat}}$.

Conj There is a flat $\mathbf{C}[\varpi]$-deformation

$$\mathrm{AH}_\varpi(\mathcal{R}(\beta)) \rightsquigarrow \mathrm{AH}(\mathcal{R}(\beta))$$
$$\mathbf{A}_{W,\varpi} \curvearrowright \mathrm{AH}_\varpi(\mathcal{R}(\beta)) \rightsquigarrow \mathbf{A}_W \curvearrowright \mathrm{AH}_0(\beta)$$

such that:

- The $\mathbf{A}_{W,\varpi}$-action is weight-filtered and degenerates to a weight-graded $\mathbf{A}_W$-action on $\mathrm{AH}_\varpi(\mathcal{R}(\beta))$.
- In the regular elliptic case, the latter extends to a $\mathbf{D}_{\nu,\varpi}^{\mathrm{rat}}$-action.

arXiv:2106.07444

Thank you for listening.