Homotopy Equivalences of Varieties Built from Braids

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Plan of this talk:

1 $\mathcal{U}$ versus U_+U_-

2 $\mathcal{U}(\beta)$ versus $\mathcal{X}(\beta\Delta^2)$

3 Traces on the Hecke Category

4 $\mathcal{T}(\beta)$ versus $\mathcal{M}_\gamma/\Lambda_\gamma$

References:

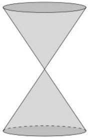
- From the Hecke Category to the Unipotent Locus (2021)
- $\mathcal{U}$ versus U_+U_- (*in progress*)
- Algebraic Braids and NAHT (*in revision*)

§1 $\mathcal{U}$ versus U_+U_-

G semisimple algebraic group

$\mathcal{U}$ unipotent locus

Ex For $G = \mathrm{SL}_2$:



Explicitly, $\mathcal{U} = \{g \in \mathrm{SL}_2 \mid \mathrm{tr}\, g = 2\}$.

Q What is $|\mathcal{U}(\mathbf{F}_p)|$ for prime $p \gg 0$?

Q Fix a Borel $B_+ \subseteq G$. How does $\mathcal{U} \cap gB_+$ vary?

$B_\pm$ opposite pair of Borels

$U_\pm$ unipotent radicals

Conj 1 At the level of $\mathbf{C}$-points,

$$\mathcal{U}_g := \mathcal{U} \cap gB_+ \quad \text{and} \quad \mathcal{V}_g := U_+U_- \cap gB_+$$

are homotopy equivalent for any g.

Thm (Steinberg '68) $|\mathcal{U}(\mathbf{F}_p)| = |U_+U_-(\mathbf{F}_p)|$.

Thm (Kawanaka '75) $|\mathcal{U}_g(\mathbf{F}_p)| = |\mathcal{V}_g(\mathbf{F}_p)|$.

Thm 1 There's an isomorphism

$$\mathrm{gr}^{\boldsymbol{w}}_* \, \mathrm{H}^{\mathrm{BM}, H_g}_*(\mathcal{U}_g) \simeq \mathrm{gr}^{\boldsymbol{w}}_* \, \mathrm{H}^{\mathrm{BM}, H_g}_*(\mathcal{V}_g),$$

where $H_g = B_+ \cap gB_+g^{-1}$. Proof uses link homology!

W Weyl group $\quad$ Br_W braid group

For each positive braid β, we'll build equivariant cartesian squares:

$$\begin{array}{ccccc} X_0(\beta, 1) & \longrightarrow & \mathcal{X}(\beta) & \longrightarrow & \mathcal{U}(\beta) \\ \downarrow & & \downarrow & & \downarrow \\ pt & \xrightarrow{B_+} & G/B_+ & \xrightarrow{1 \times \mathrm{id}} & \mathcal{U} \times G/B_+ \end{array}$$

where $X_0(\beta, 1)$ is the *braid variety* of Mellit, CGGS.

Prop If $B_+ \xrightarrow{w} gB_+g^{-1}$ and $\beta = \sigma_w$, then

$$[\mathcal{U}_g/H_g] \simeq [\mathcal{U}(\beta)/G],$$
$$[\mathcal{V}_g/H_g] \approx [\mathcal{X}(\beta\Delta^2)/G],$$

where $\Delta = \sigma_{w_0}$ is the half twist.

§2 $\mathcal{U}(\beta)$ versus $\mathcal{X}(\beta\Delta^2)$

$\sigma_w \in Br_W$ is the minimal positive lift of $w \in W$.

Suppose $\beta = \sigma_{w_1} \cdots \sigma_{w_k}$. Then:

$$\mathcal{U}(\beta) = \left\{ (u, B_1, \ldots, B_k) \;\middle|\; \begin{array}{l} B_{i-1} \xrightarrow{w_i} B_i, \\ B_0 = uB_k u^{-1} \end{array} \right\}$$

$$\mathcal{X}(\beta) = \text{subvariety of } \mathcal{U}(\beta) \text{ where } u = 1$$

$$X_0(\beta, 1) = \text{subvariety of } \mathcal{X}(\beta) \text{ where } B_k = B_+$$

Ex For $\mathbf{1} := \sigma_{\mathrm{id}}$, we have:

$$\mathcal{U}(\mathbf{1}) = \{(u, B_1) \mid u \in B_1\} = \text{Springer resolution}$$

$$\mathcal{X}(\mathbf{1}) \simeq G/B_+$$

Suppose $B_+ \xrightarrow{w} gB_+g^{-1}$ and $H_g = B_+ \cap gB_+g^{-1}$.

Let $O_w = \{(B', B) \mid B' \xrightarrow{w} B\} \simeq G/H_g$.

1 $\mathcal{U}_g = \mathcal{U} \cap gB_+$ is the fiber of

$$\mathcal{U}(\sigma_w) \xrightarrow{(B_1,\, uB_1u^{-1})} O_w$$

above (B_+, gB_+g^{-1}). Thus $[\mathcal{U}_g/H_g] \simeq [\mathcal{U}(\sigma_w)/G]$.

2 $\mathcal{V}_g = U_+U_- \cap gB_+$ bijects onto the fiber of

$$\mathcal{X}(\sigma_w\Delta^2) \xrightarrow{(B_1,\, B_3)} O_w$$

above (B_+, gB_+g^{-1}). Thus $[\mathcal{V}_g/H_g] \approx [\mathcal{X}(\sigma_w\Delta^2)/G]$.

Thm 2 For any positive β, we have

$$\mathrm{gr}_*^{\boldsymbol{w}}\,\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta)) \simeq \mathrm{gr}_*^{\boldsymbol{w}}\,\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{X}(\beta\Delta^2)).$$

Pf sketch Using Springer theory, we'll show

$$\mathrm{gr}_*^{\boldsymbol{w}}\,\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta)) \simeq [a^{|\beta|-r}]\,\mathrm{HHH}(\hat{\beta}),$$
$$\mathrm{gr}_*^{\boldsymbol{w}}\,\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{X}(\beta)) \simeq [a^{|\beta|+r}]\,\mathrm{HHH}(\hat{\beta}),$$

where HHH is *triply-graded Khovanov–Rozansky homology* and $r = \mathrm{rk}(W)$.

GHMN proved that for any braid β,

$$[a^{|\beta|-r}]\,\mathrm{HHH}(\hat{\beta}) \simeq [a^{|\beta|+r}]\,\mathrm{HHH}(\widehat{\beta\Delta^2}).$$

Thm 1 ($\mathcal{U}_g$ vs $\mathcal{V}_g$) is a special case of **Thm 2**, which in turn will be a corollary of a stronger result.

The *Steinberg variety* of β is

$$\mathcal{Z}(\beta) = \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta).$$

Via pull-push functors,

$$\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\mathbf{1}))$$

forms an algebra that acts on $\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\beta))$.

Thm (Lusztig '88) As algebras,

$$\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\mathbf{1})) \simeq \mathbf{C}[W] \ltimes \mathrm{Sym}(\mathbf{t}),$$

where $\mathbf{t}$ is the Cartan algebra of $\mathbf{g} = \mathrm{Lie}(G)$.

Thm 3 For any positive β and $0 \leq k \leq r$, we have

$$[\Lambda^k(\mathbf{t})]\;\mathrm{gr}_*^{\boldsymbol{w}}\,\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\beta)) \simeq [a^{|\beta|-r+2k}]\,\mathrm{HHH}(\hat{\beta}).$$

§3 Traces on the Hecke Category

Thm 3 relies on work of Webster–Williamson.

Recall that Khovanov–Rozansky homology is really a *monoidal trace functor*

$$\mathrm{HHH} : \mathbf{H}_W \to \mathbf{Vect}_3,$$

where $\mathbf{H}_W = \mathrm{K}^b(\mathbf{SBim}_W)$ is the *Hecke category*.

Rouquier gave a "strict" categorification of Br_W:

$$\beta \mapsto \mathcal{R}(\beta)$$

such that $\mathrm{HHH}(\hat{\beta}) \propto \mathrm{HHH}(\mathcal{R}(\beta))$.

Webster–Williamson constructed this functor from the geometry of "mixed sheaves".

Bruhat decomposition: $G = \coprod_{w \in W} B_+ w B_+$

Let IC_w be the perverse sheaf formed by $!*$-extension of the constant sheaf along $BwB \hookrightarrow G$.

Soergel essentially matched $\mathbf{SBim}_W$ with

$$\left\langle \mathrm{IC}_w\langle m\rangle \;\middle|\; \begin{matrix} w \in W \\ m \in \mathbf{Z} \end{matrix} \right\rangle_{\oplus} \subseteq \mathrm{D}^b_m(B_+\backslash G/B_+),$$

where D^b_m is the *mixed* derived category and $\langle 1\rangle$ is the *shift-twist*.

Roughly, pull-push through the *horocycle diagram*

$$[B_+\backslash G/B_+] \leftarrow [G/B_{+,\,\mathrm{Ad}}] \to [G/G_{\mathrm{Ad}}]$$

categorifies the cocenter map of the Hecke algebra.

The functor

$$\mathrm{CH} : \mathrm{D}^b_m(B_+\backslash G/B_+) \to \mathrm{D}^b_m(G/G_{\mathrm{Ad}})$$

induces a monoidal trace on $\mathbf{H}_W$.

Thm (WW) For all $w \in W$, we have

$$\mathrm{gr}^{\boldsymbol{w}}_{i+j}\, \mathrm{H}^j_G(G, \mathrm{CH}(\mathrm{IC}_w)) \simeq \mathrm{HH}^i(\mathrm{H}^{i+j}_{B\times B}(G, \mathrm{IC}_w)),$$

where HH^* is Hochschild homology over $\mathrm{Sym}(\mathbf{t})$.

So HHH factors as

$$\mathbf{H}_W \xrightarrow{\mathrm{CH}} \mathrm{K}^b(\mathbf{C}_G) \xrightarrow{\mathrm{gr}^{\boldsymbol{w}}_*\, \mathrm{H}^*_G} \mathbf{Vect}_3,$$

where $\mathbf{C}_G = \langle \mathrm{CH}(\mathrm{IC}_w)\langle m\rangle : w, m\rangle_\oplus \subseteq \mathrm{D}^b_m(G/G_{\mathrm{Ad}})$.

Want to assemble this into the homology of a variety.

Let $i : \mathcal{U} \to G$ be the inclusion. We build:

$$\begin{array}{ccccc}
\mathrm{K}^b(\mathbf{SBim}_W) & \xrightarrow{\mathrm{CH}} & \mathrm{K}^b(\mathbf{C}_G) & \xrightarrow{\iota^*} & \mathrm{K}^b(\mathbf{C}_{\mathcal{U}}) \\
\rho \downarrow & & & & \downarrow \rho \\
\mathrm{D}^b_m(B_+\backslash G/B_+) & \xrightarrow{\mathrm{CH}} & \mathrm{D}^b_m(G/G) & \xrightarrow{\iota^*} & \mathrm{D}^b_m(\mathcal{U}/G)
\end{array}$$

The ρ are *weight realization functors*. Their existence uses an Ext-vanishing condition that fails for $\mathbf{C}_G$.

Let $\mathcal{S} \in \mathrm{D}^b_m(\mathcal{U}/G)$ be the (mixed) Springer sheaf.

Lem As contravariant functors on $\underline{\mathbf{C}}_G$,

$$(\mathrm{gr}^{\boldsymbol{w}}_{i+j}\, \mathrm{H}^j_G(G, -))^\vee \propto [\Lambda^i(\mathbf{t})]\, \underline{\mathrm{Hom}}^0(i^*(-), \mathcal{S}\langle j\rangle).$$

Can check on summands of the Grothendieck sheaf.

Pf sketch of Thm 3 Chasing $\mathcal{R}(\beta)$ through the upper-right part and applying

$$(\clubsuit) \qquad \bigoplus_{i,j} [\Lambda(\mathbf{t})]\, \underline{\mathrm{Hom}}^0(-, \mathcal{S}\langle j\rangle[i-j]_\triangle),$$

we recover $\mathrm{HHH}(\hat{\beta})^\vee$, by **Thm (WW)** and **Lem**.

Chasing it through the lower-left and applying

$$(\spadesuit) \qquad \bigoplus_{i,j} [\Lambda(\mathbf{t})]\, \mathrm{gr}_j^{\boldsymbol{w}}\, \underline{\mathrm{Hom}}^i(-, \mathcal{S}),$$

we recover $[\Lambda(\mathbf{t})]\ \mathrm{gr}_*^{\boldsymbol{w}}\ \mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\beta))$.

Finally, match $(\clubsuit)$ and $(\spadesuit)$ using "purity" in $\mathbf{C}_{\mathcal{U}}$.

Slogan: Difference between Springer and Grothendieck is homology of a maximal torus, *i.e.*, $\Lambda(\mathbf{t})$.

Ex Take $G = \mathrm{SL}_2$ and $W = S_2$ and $\beta = \sigma^3$.

Here, $\hat{\beta}$ is a trefoil and

$$\begin{aligned} &\dim_{a,q,t} \mathrm{HHH}(\hat{\beta}) \\ &= a^2(q^{-1} + qt^2) + a^4t^3 \\ &= (at)^{|\beta|} a^{-r} (q^{-1}t^{-3} + qt^{-1} - (a^2 q^{\frac{1}{2}}\, t) q^{-\frac{1}{2}} t^{-1}). \end{aligned}$$

The tuples $(0, -2, 3)$, $(0, 2, 1)$, $(1, -1, 1)$ correspond to

$$\mathrm{gr}_0^{\boldsymbol{w}}\, \mathrm{H}_2^{\mathrm{BM},G}, \quad \mathrm{gr}_4^{\boldsymbol{w}}\, \mathrm{H}_4^{\mathrm{BM},G}, \quad \mathrm{gr}_2^{\boldsymbol{w}}\, \mathrm{H}_2^{\mathrm{BM},G}.$$

The red term is the difference between $\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta))$ and $\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}(\beta))$.

Note that $[\mathcal{U}(\beta)/G]$ retracts onto a 2-sphere.

§4 $\mathcal{M}_\gamma/\Lambda_\gamma$ versus $\mathcal{T}(\beta)$

Recall the *affine Grassmannian*:

$$\mathcal{M}(\mathbf{C}) = G(\mathbf{C}((x)))/G(\mathbf{C}[\![x]\!])$$

Any $\gamma \in \mathbf{g}(\mathbf{C}((x)))$ defines a vector field on $\mathcal{M}$.

If γ is regular semisimple, then the fixed-point locus

$$\mathcal{M}_\gamma = \{[g] \in \mathcal{M} \mid \mathrm{Ad}(g^{-1})\gamma \in \mathbf{g}(\mathbf{C}[\![x]\!])\}$$

is a finite-dimensional ind-scheme called the *affine Springer fiber* of γ.

Note that $G_\gamma := G(\mathbf{C}((x)))_\gamma \curvearrowright \mathcal{M}_\gamma$. Let

$$\Lambda_\gamma = \mathrm{Im}(\lambda \mapsto x^\lambda : \mathbf{X}_*(G_\gamma) \to G_\gamma).$$

Kazhdan–Lusztig show $\mathcal{M}_\gamma/\Lambda_\gamma$ is a projective variety.

Let $\mathbf{c} = \mathbf{g} /\!\!/ G \simeq \mathbf{t} /\!\!/ W$, the *Chevalley quotient*.

If $\mathbf{g} \to \mathbf{c}$ sends $\gamma \mapsto a \in \mathbf{c}(\mathbf{C}[\![x]\!])$, then the $\simeq$ type of

$$(\mathcal{M}_\gamma, \Lambda_\gamma)$$

only depends on a.

At the same time, a defines an infinitesimal loop in $\mathbf{c}^{\mathrm{reg}} = \mathbf{t}^{\mathrm{reg}} /\!\!/ W$. Turns out to give a conjugacy class

$$[\beta] \subseteq Br_W.$$

How does the topology of $\mathcal{M}_\gamma$ depend on $[\beta]$?

Conj 1 If γ is *nil-elliptic* ($\implies$ $\hat{\beta}$ is a knot), then $[\mathcal{U}(\beta)/G]$ essentially deformation retracts onto $\mathcal{M}_\gamma$.

Special case of a more general conjecture.

In general: Suppose $\beta = \sigma_{s_1} \cdots \sigma_{s_k}$ with s_i's simple,

$$\beta \mapsto w \in W.$$

The rank of Λ_γ equals $\dim \mathbf{t}^w$.

Fix $B = T \ltimes U \subseteq G$ and a lift $W \to N(T)$. Let

$$m_\beta : (T^w)^\circ \times \dot{s}_{i_1} U_{\alpha_{i_1}} \times \cdots \times \dot{s}_{i_k} U_{\alpha_{i_k}} \to G$$

be the multiplication map, and let $\mathcal{T}(\beta) = m_\beta^{-1}(\mathcal{U})$.

When β contains Δ as a prefix, there is a T^w-bundle

$$\mathcal{T}(\beta) \to [\mathcal{U}(\beta)/G].$$

Expect $\mathcal{T}(\beta)$ may be related to *y-ified* HHH, just as $[\mathcal{U}(\beta)/G]$ is related to usual HHH.

Conj 2 For arbitrary γ such that $\gamma(0) = 0$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\beta) & \xrightarrow{\textit{def. retract}} & \mathcal{M}_\gamma/\Lambda_\gamma \\ \downarrow & & \downarrow \\ [\mathcal{U}/G] & = & [\mathcal{N}/G] \end{array}$$

Moreover, the retraction identifies:

halved weight filtration on $\mathcal{T}(\beta)$
$\simeq$
perverse filtration on $\mathcal{M}_\gamma/\Lambda_\gamma$

View $\mathcal{T}(\beta)$ and $\mathcal{M}_\gamma/\Lambda_\gamma$ as Betti and Dolbeault sides of a nonabelian Hodge correspondence.

Thm 4 Evidence for diagram at level of q-deformed Euler characteristics, for *equivalued* elliptic γ.

Ex Take $G = \mathrm{SL}_2$ and $S_2 = \langle s \rangle$ and $Br_2 = \langle \sigma \rangle$.

If $\beta = \sigma^2$, then $w = 1$ and $(T^w)^\circ = T$.

$$\begin{aligned} \mathcal{T}(\beta) &= \left\{(a, z_1, z_2) : \mathrm{tr} \left(\begin{smallmatrix} a & \\ & \frac{1}{a} \end{smallmatrix}\right) \left(\begin{smallmatrix} & -1 \\ 1 & z_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} & -1 \\ 1 & z_2 \end{smallmatrix}\right) = 2\right\} \\ &= \{(a, z_1, z_2) \in \mathbf{G}_m \times \mathbf{A}^2 : z_1 z_2 = (1+a)^2\} \end{aligned}$$

deformation retracts onto a pinched torus.

If $\beta = \sigma^3$, then $w = s$ and $(T^w)^\circ = 1$.

$$\begin{aligned} \mathcal{T}(\beta) &= \left\{(z_1, z_2, z_3) : \mathrm{tr} \left(\begin{smallmatrix} & -1 \\ 1 & z_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} & -1 \\ 1 & z_2 \end{smallmatrix}\right) \left(\begin{smallmatrix} & -1 \\ 1 & z_3 \end{smallmatrix}\right) = 2\right\} \\ &= \{(z_1, z_2, z_3) \in \mathbf{A}^3 : z_1 z_2 z_3 = 2 + z_1 + z_2 + z_3\} \end{aligned}$$

deformation retracts onto a sphere.

Thank you for listening.