Affine Springer Fibers and Level-Rank Duality

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1 Springer Theory

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Mainly about joint work with Ting Xue:

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See also the extended abstract on my website, which we have submitted to FPSAC '25.

1 Springer Theory

Work over $\mathbf{C}$.

$\mathbf{G}$	connected reductive group
$\mathbf{A}$	maximal torus
W	Weyl group

The *rational Cherednik algebra* D_c^{rat} is a deformation of $\mathbf{C}W \ltimes \mathcal{D}(\mathbf{a})$ depending on a parameter $c \in \mathbf{C}$.

D_c^{rat}	$\mathrm{U}\mathbf{g}$
$\mathbf{C}[\mathbf{a}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathbf{a}^*]$	$\mathrm{U}\mathbf{n}_- \otimes \mathrm{U}\mathbf{a} \otimes \mathrm{U}\mathbf{n}_+$
$\Delta_c(\chi)$	$\Delta(\lambda)$
$L_c(\chi)$	$L(\lambda)$

For *c rational*, D_c^{rat} can fail to be semisimple. This is the most interesting case.

For c rational and *positive*, D_c^{rat}-modules from the geometry of *affine Springer fibers*.

$\mathbf{B}$	Borel containing $\mathbf{A}$
$\mathbf{I} \subseteq \mathbf{G}[\![z]\!]$	Iwahori lifting $\mathbf{B} \subseteq \mathbf{G}$

The affine Springer fiber over $\gamma \in \mathbf{g}(\!(z)\!)$ is

$$\mathcal{F}l_\gamma = \{g\mathbf{I} \in \mathbf{G}(\!(z)\!)/\mathbf{I} \mid \gamma \in \mathrm{Lie}(g\mathbf{I}g^{-1})\}.$$

Note that $\mathbf{G}(\!(z)\!)/\mathbf{I}$ is infinite-dimensional.

We say that γ is *regular semisimple* iff $\mathbf{G}(\!(z)\!)_\gamma^\circ$ is a maximal torus.

Here $\mathcal{F}l_\gamma$ is finite-dimensional!

But it varies wildly over $\mathbf{g}(\!(z)\!)^{\mathrm{rs}} \subseteq \mathbf{g}(\!(z)\!)$.

Fix rational $c = \frac{d}{m} > 0$ in lowest terms.

Let $\mathbf{C}^\times \curvearrowright \mathbf{G}(\!(z)\!)$ according to

$$\boxed{c \cdot g(z) = \mathrm{Ad}(c^{d\rho^\vee})g(c^m z).} \quad \left(\rho^\vee = \textstyle\sum_\alpha \omega_\alpha^\vee\right)$$

(Oblomkov–Yun) $\mathcal{F}l_\gamma$ is locally constant over

$$\mathbf{g}_{d/m}^{\mathrm{rs}} = \{\gamma \in \mathbf{g}(\!(z)\!)^{\mathrm{rs}} \mid c \cdot \gamma = c^d \gamma\},$$

and $\mathbf{C}^\times \curvearrowright \mathcal{F}l_\gamma$ for such γ.

We say that γ is *homogeneous of slope* $\frac{d}{m}$.

Example Take $\mathbf{G} = \mathbf{SL}_2$ and $\mathbf{B}$ upper-triangular.

Then $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ z & \end{pmatrix}, \begin{pmatrix} z & \\ & -z \end{pmatrix}$ have slopes $0, \frac{1}{2}, 1$.

(Oblomkov–Yun) Take $\mathbf{G}$ simply-connected, simple.

For $\gamma \in \mathbf{g}^{\mathrm{rs}}_{d/m}$ such that $\mathcal{F}l_\gamma$ is proper:

- A *perverse filtration* $\mathsf{P}_{\leq *}$ on $\mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)$.

It arises from a Ngô-type global model.

- An action of $D^{\mathrm{rat}}_{d/m}$ on

$$\mathcal{E}_\gamma := \mathrm{gr}^{\mathsf{P}}_* \, \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(\mathbf{G}_{0,\gamma})}|_{\epsilon \to 1},$$

where $\mathbf{G}_0 = (\mathbf{G}(\!(z)\!)^{\mathbf{C}^\times})^\circ$ and $\epsilon \in \mathrm{H}^2_{\mathbf{C}^\times}(\mathit{point})$.

As a module, $\mathcal{E}_\gamma$ contains $L_{d/m}(\chi_{\mathsf{triv}})$.

Equality holds when m is the Coxeter number.

Problem Give a formula for $D^{\mathrm{rat}}_{d/m} \curvearrowright \mathcal{E}_\gamma$ in general.

In practice, too hard. Replace with

$$E_\gamma := \sum_i (-1)^i \, \mathrm{gr}^{\mathsf{P}}_* \, \mathrm{H}^i_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(\mathbf{G}_{0,\gamma})}|_{\epsilon \to 1}.$$

Idea $D^{\mathrm{rat}}_{d/m}$ commutes with monodromy of $\mathcal{E}_\gamma$ over

$$\mathbf{c}^{\mathrm{rs}}_{d/m} \subseteq \mathbf{g}^{\mathrm{rs}}_{d/m},$$

a Kostant-type transverse slice to $\mathbf{G}_0 \curvearrowright \mathbf{g}^{\mathrm{rs}}_{d/m}$.

The monodromy seems to factor through an algebra from *Deligne–Lusztig theory*.

Deligne–Lusztig studied groups over *finite fields*. But up to Tate twist,

$$\mathrm{Gal}(\bar{\mathbf{F}}_q|\mathbf{F}_q) \quad \simeq \quad \hat{\mathbf{Z}} \quad \simeq \quad \mathrm{Gal}(\overline{\mathbf{C}(\!(z)\!)}|\mathbf{C}(\!(z)\!)).$$

(Outer) forms of $\mathbf{G}$ are classified by Dynkin automorphisms in the same way over $\mathbf{F}_q$ as over $\mathbf{C}(\!(z)\!)$.

Much of Oblomkov–Yun's setup generalizes from $\mathbf{G}$ to any of its forms $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$.

The tori $\mathbf{A}, \mathbf{G}_\gamma$ generalize to forms $\mathbf{A}_{\mathbf{C}(\!(z)\!)}, \mathbf{G}_{\mathbf{C}(\!(z)\!),\gamma}$.

These have corresponding forms $\mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q}$.

2 Deligne–Lusztig Theory

Work over $\bar{\mathbf{F}}_q$ for good q.

$$\{\text{forms of } \mathbf{G} \text{ over } \mathbf{F}_q\} \quad \leftrightarrow \quad \{\text{Frobenii } F \curvearrowright \mathbf{G}\}$$

We say that $G = \mathbf{G}^F$ is a *finite group of Lie type*.

F-stable Levis $\mathbf{L} \subseteq \mathbf{G}$ correspond to Levis $L \subseteq G$.

Deligne–Lusztig introduced varieties† $Y_{\mathbf{L}}^{\mathbf{G}}$ such that

$$G \quad \curvearrowright \quad \mathrm{H}_c^*(Y_{\mathbf{L}}^{\mathbf{G}}) \quad \curvearrowleft \quad L.$$

Induction map $R_L^G : \mathrm{K}_0(L) \to \mathrm{K}_0(G)$:

$$R_L^G(\lambda) = \sum_i (-1)^i \mathrm{H}_c^i(Y_{\mathbf{L}}^{\mathbf{G}})[\lambda].$$

† Actually, $Y_{\mathbf{L}}^{\mathbf{G}}$ depends on a parabolic $\mathbf{P} \supseteq \mathbf{L}$.

(Broué–Malle) For m-*regular* maximal tori $\mathbf{T}$, a specific algebra $H_T^G(\mathsf{q})$ such that

$$H_T^G(\zeta_m) = \bar{\mathbf{Q}} W_T^G, \quad \text{where } W_T^G = N_G(T)/T.$$

They conjecture:

1 $H_T^G(q) \otimes \bar{\mathbf{Q}}_\ell \simeq \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{T}}^{\mathbf{G}})[1_T])$.

2 As a virtual $(G, H_T^G(q))$-bimodule,

$$R_T^G(1_T) = \sum_{\substack{\rho \in \mathrm{Irr}(G) \\ (\rho, R_T^G(1_T)) \neq 0}} \varepsilon_{T,\rho}(\rho \otimes \chi_{T,\rho,q})$$

where $\varepsilon_{T,\rho} \in \{\pm 1\}$ and $\chi_{T,\rho} \in \mathrm{Irr}(W_T^G)$.

(And $\chi_{T,\rho,q} \in \mathrm{K}_0(H_T^G(q))$ corresponds to $\chi_{T,\rho}$.)

Back to Springer. $(\mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q} \leftrightarrow \mathbf{A}_{\mathbf{C}(\!(z)\!)}, \mathbf{G}_{\mathbf{C}(\!(z)\!),\gamma})$

It turns out that $\mathbf{A}$ and $\mathbf{T}$ are 1- and m-regular.

Moreover, $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}})$ is the braid group of W_T^G.

Conjecture (T–Xue)

1 $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}}) \curvearrowright \mathcal{E}_\gamma$ factors through $H_T^G(1)$.

2 As a virtual $(D_{d/m}^{\mathrm{rat}}, H_T^G(1))$-bimodule,†

$$E_\gamma = \sum_{\substack{\rho \in \mathrm{Irr}(G) \\ (\rho, R_A^G(1_A)) \neq 0 \\ (\rho, R_T^G(1_T)) \neq 0}} \varepsilon_{T\rho}(\Delta_{d/m}(\chi_{A,\rho}) \otimes \chi_{T,\rho,1}).$$

† In general, $D_{d/m}^{\mathrm{rat}}$ is defined using W_A^G.

Theorem (T–Xue) True in these cases:

- m is the (twisted) Coxeter number of $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$.
- $(\mathbf{G}_{\mathbf{C}(\!(z)\!)}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.

Under a conjecture of OY, true in further cases.

Example Take $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$ split, m its Coxeter number.

$\chi_{A,\rho}$ runs over characters $\chi_{\wedge^k(\mathbf{a})}$ of W_A^G.

$\chi_{T,\rho}$ runs over *all* characters of $W_T^G = \mathbf{Z}/m\mathbf{Z}$.

In $\mathrm{K}_0(D_{d/m}^{\mathrm{rat}})$,

$$[E_\gamma] = \sum_k (-1)^k [\Delta_{d/m}(\chi_{\wedge^k(\mathbf{a})})]$$
$$= [L_{d/m}(\chi_{\mathsf{triv}})].$$

Cf. the BGG resolution of Berest–Etingof–Ginzburg.

3 Level-Rank Duality

Compare E_γ given by

$$\sum_\rho \varepsilon_{T,\rho}(\Delta_{d/m}(\chi_{A,\rho}) \otimes \chi_{T,\rho,1})$$

with $R_A^G(1_A) \otimes_{\bar{\mathbf{Q}}_\ell G} R_T^G(1_T)$ given by

$$\sum_\rho \varepsilon_{T,\rho}(\chi_{A,\rho,q} \otimes \chi_{T,\rho,q}).$$

The *Knizhnik–Zamolodchik functor*

$$\mathsf{KZ} : \mathsf{Rep}(D_{d/m}^{\mathrm{rat}}) \to \mathsf{Rep}(H_A^G(\zeta_m))$$

sends $\mathsf{KZ}(\Delta_{d/m}(\chi)) = \chi_{\zeta_m}$. Thus an analogy:

$$\boxed{\mathbf{F}_q \;:\; (q, q) \;::\; \mathbf{C}(\!(z)\!) \;:\; (\zeta_m, 1)}$$

The symmetry between A and T led us to new discoveries about the Harish–Chandra theory of G.

Let $\mathrm{Uch}(G)$ be the set of *unipotent* irreps of G, which occur in $R_T^G(1_T)$ for some maximal torus $\mathbf{T}$.

(Broué–Malle–Michel) Fix a positive integer l.

- $\mathbf{L} \subseteq \mathbf{G}$ is *l-split* iff $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{S})^\circ$, where

  $\mathbf{S}$ is a torus with $|S|$ a power of $\Phi_l(q)$.

- $\lambda \in \mathrm{Uch}(L)$ is *l-cuspidal* iff $(\lambda, R_M^G(\mu)) = 0$ for any l-split $M \neq L$.

As we run over pairs $(\mathbf{L}, \lambda)$ up to conjugacy,

$$\mathrm{Uch}(G) = \coprod \mathrm{Uch}(G)_{\mathbf{L},\lambda},$$

where $\mathrm{Uch}(G)_{\mathbf{L},\lambda} = \{\rho \mid (\rho, R_L^G(\lambda)) \neq 0\}$.

For $l = 1$, these are classical *Harish-Chandra series*.

Generalizing our discussion for maximal tori:

Broué–Malle define a Hecke algebra $H_{L,\lambda}^G(\mathsf{q})$ such that

$$H_{L,\lambda}^G(\zeta_l) = \bar{\mathbf{Q}}W_{L,\lambda}^G, \text{ where } W_{L,\lambda}^G = C_{N_G(L)/L}(\lambda).$$

They conjecture:

1 $H_{L,\lambda}^G(q) \otimes \bar{\mathbf{Q}}_\ell = \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{L}}^{\mathbf{G}})[\lambda])$.

2 As a virtual $(G, H_{L,\lambda}^G(q))$-bimodule,

$$R_L^G(\lambda) = \sum_{\rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda}} \varepsilon_{L,\lambda,\rho}(\rho \otimes \chi_{L,\lambda,\rho,q})$$

where $\varepsilon_{L,\lambda,\rho} \in \{\pm 1\}$ and $\chi_{L,\lambda,\rho} \in \mathrm{Irr}(W_{L,\lambda}^G)$.

Via the *decomposition map*

$$\chi \mapsto \chi_{\zeta_m} : \mathrm{Irr}(W^G_{L,\lambda}) \to \mathrm{K}_0(H^G_{L,\lambda}(\zeta_m)),$$

we partition $\mathrm{Irr}(W^G_{L,\lambda})$ into *blocks*, describing how $H^G_{L,\lambda}(\zeta_m)$ fails to be semisimple.

Conjecture (T–Xue) Fix l, m.

Fix an l-cuspidal $(\mathbf{L}, \lambda)$ and m-cuspidal $(\mathbf{M}, \mu)$.

1 The set

$$\{\chi_{L,\lambda,\rho} \mid \rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(G)_{\mathbf{M},\mu}\},$$
$$\textit{resp.} \quad \{\chi_{M,\mu,\rho} \mid \rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(G)_{\mathbf{M},\mu}\},$$

is a union of $H^G_{L,\lambda}(\zeta_m)$-, *resp.* $H^G_{M,\mu}(\zeta_l)$-blocks.

2 The indexing induces a matching of blocks.

Theorem (T–Xue) (1), (2) are compatible with block sizes for essentially all G, l, m with G exceptional.

Conjecture (T–Xue) In the preceding setup:

3 Via KZ functors, the bijection in (2) lifts to a derived equivalence between category-O blocks of appropriate rational Cherednik algebras.

Theorem (T–Xue) (1), (2), (3) hold for $G = \mathrm{GL}_n$ when l, m are coprime.

Note that $W_{L,\lambda}^{\mathrm{GL}_n} \simeq S_N \ltimes \mathbf{Z}_l^N$ for some N, *etc.*

$$\mathsf{Rep}(H_{L,\lambda}^{\mathrm{GL}_n}(\zeta_m)) \quad \text{and} \quad \mathsf{Rep}(H_{M,\mu}^{\mathrm{GL}_n}(\zeta_l))$$

can be interpreted in terms of *higher-level Fock spaces*

$$\bigoplus_{\substack{\vec{s} \in \mathbf{Z}^l \\ |\vec{s}| = s}} \Lambda_{\mathsf{q}}^{\vec{s}} \xleftarrow{\sim} \Lambda_{\mathsf{q}}^{s} \xrightarrow{\sim} \bigoplus_{\substack{\vec{r} \in \mathbf{Z}^m \\ |\vec{r}| = s}} \Lambda_{\mathsf{q}}^{\vec{r}}.$$

Above, $\Lambda_{\mathsf{q}}^{\vec{s}} \simeq \bigoplus_N \mathrm{K}_0(S_N \ltimes \mathbf{Z}_l^N) \otimes \mathbf{Q}(\mathsf{q})$, *etc.*

Level-rank duality of Frenkel, Uglov, Chuang–Miyachi, Rouquier–Shan–Varagnolo–Vasserot…

Our conjectures generalize level-rank duality from GL_n to arbitrary G.

Thank you for listening.