Regular Elements as Twisted Principal Series

Minh-Tâm Quang Trinh

Howard University

1 Centralizers

2 Combinatorics

3 Correspondences

1 Centralizers

Let G and H be reductive groups over a local field.

In Langlands, *functoriality* is the principle that a map ${}^L H \to {}^L G$ induces a *transfer*

$$\left\{ \begin{array}{c} \text{admissible irreps} \\ \text{of } H \end{array} \right\} \to \left\{ \begin{array}{c} \text{admissible irreps} \\ \text{of } G \end{array} \right\}.$$

One of the basic testing grounds was and is *endoscopy*.

Roughly, H is *endoscopic* to G iff ${}^L H = C_{{}^L G}(s)^\circ$ for some semisimple element $s \in {}^L G$.

Today's story is **not** an analogue of endoscopy for Weyl groups, but...

Let W be a well-generated (complex) reflection group.

(Springer '74)

An element $w \in W$ is *regular* iff it has a regular eigenvector in the reflection rep V.

In this case, $C_W(w)$ is also a reflection group.

Example The identity e.

Example For W real, the longest element $w_\circ$.

Example For W irreducible, Coxeter elements.

(Broué–Michel '97) In Weyl groups:

w is regular iff its minimal positive braid lift has a central power.

Take W the Weyl group of a finite reductive group

$$G = \mathbf{G}^F, \quad \text{where } F \text{ is a } q\text{-Frobenius map.}$$

A choice of F-stable, maximally F-split $\mathbf{A} \subseteq \mathbf{G}$ yields:

$$\begin{aligned} \{\text{maximal tori of } G\}/\text{conj.} &\simeq W/\text{conj.} \\ T &= \mathbf{A}^{wF} \end{aligned}$$

(Deligne–Lusztig '76)

All irreps of G arise from such T via *cohomological** induction maps

$$R_T^G \colon K_0(T) \to K_0(G).$$

Special features when T is of type $[w]$ for *regular* w.

* say, over $\mathbf{K} = \bar{\mathbf{Q}}_\ell$.

An irrep of G is *unipotent* iff it belongs to

$$\mathrm{Uch}_T^G = \{\rho \mid (\rho, R_T^G(1))_G \neq 0\} \quad \text{for some } T.$$

For such ρ, a *generic degree* $\mathrm{Deg}_\rho(x) \in \mathbf{Q}[x]$ such that

$$\boxed{\deg \rho = \mathrm{Deg}_\rho(q).}$$

(Lusztig, Broué–Malle–Michel '93)

If w is regular of order m, and T is of type $[w]$, then

$$\mathrm{Uch}_T^G = \{\text{unipotent } \rho \mid \mathrm{Deg}_\rho(\zeta_m) \neq 0\}.$$

And $\exists\, \chi_T^G \colon \mathrm{Uch}_T^G \xrightarrow{\sim} \mathrm{Irr}(C_W(w))$ such that

$$\boxed{\deg \chi(\rho) = |\mathrm{Deg}_\rho(\zeta_m)|.}$$

Proved case by case. Morally, why the bijection?

Conjecture (Broué–Malle '93) $H_T^G(x)$ such that

$$\boxed{\begin{aligned} H_T^G(q) &\simeq \mathrm{End}_G(\mathrm{H}^*_{\mathrm{c,ét}}(X_w, \mathbf{K})), \\ H_T^G(\zeta_m) &\simeq \mathbf{K}C_W(w). \end{aligned}}$$

χ_T^G should arise from a double centralizer theorem.

Example Write $A := \mathbf{A}^F$. It is of type $[e]$.

$$X_e = (\mathbf{G}/\mathbf{B})^F = G/B.$$

$R_A^G(1)$ consists of the *unipotent principal series*: the irreps occuring in $\mathbf{C}[G/B]$.

$H_A^G(x)$ is the usual Hecke algebra of W.

Example Take $G = \mathrm{Sp}_4(\mathbf{F}_q)$ and T of *Coxeter type*.

Any Coxeter element c has order $h = 4$.

	$\mathrm{Deg}_\rho(x)$	$\mathrm{Deg}_\rho(1)$	$\mathrm{Deg}_\rho(\zeta_4)$
	1	1	1
	$\frac{1}{2}x(x^2+1)$	1	$\cdot$
	$\frac{1}{2}x(x+1)^2$	2	-1
	$\frac{1}{2}x(x^2+1)$	1	$\cdot$
	x^4	1	1
θ_{10} $\rightarrow$	$\frac{1}{2}x(x-1)^2$	$\cdot$	1

$W = C_W(e)$ is dihedral of order 8, while $C_W(c)$ is cyclic of order 4.

Anything special about $\mathrm{Uch}_A^G \cap \mathrm{Uch}_T^G$?

(≈ Lusztig '78) For G almost simple and $\rho \in \mathrm{Uch}_A^G$,

$$\mathrm{Deg}_\rho(\zeta_h) = \begin{cases} (-1)^k & \chi_A^G(\rho) = \Lambda^k(V), \\ 0 & \text{else.} \end{cases}$$

So for T of Coxeter type, χ_A^G restricts to

$$\mathrm{Uch}_A^G \cap \mathrm{Uch}_T^G \xrightarrow{\sim} \{\Lambda^k(V) \mid 0 \le k \le r\},$$

where $r = \mathrm{rank}(G)$.

For $W = S_n$, indirectly recovers Frobenius's result:

$$\chi(n\text{-cycle}) = \begin{cases} (-1)^k & \chi = \Lambda^k(V), \\ 0 & \text{else.} \end{cases}$$

2 Combinatorics

$\mathrm{Uch}_A^G \cap \mathrm{Uch}_T^G$ is implicit throughout my work with Nathan.

For any word $\vec{s}$ in simple reflections, write

$$\vec{w} \nearrow \vec{s} \quad \text{iff} \quad \vec{w} \text{ is a } \textit{Deodhar walk} \text{ for } \vec{s}.$$

This means a walk in W, starting and ending at e, such that for all i:

- either $w_{i+1} = w_i s_{i+1}$ or $w_{i+1} = w_i$.
- $w_{i+1} \leq w_i s_{i+1}$. *If you can descend, you must.*

Let $d_{\vec{w}}$, *resp.* $e_{\vec{w}}$, count the descents, *resp.* stays.

These appear in point counts of Deodhar cells.

(Galashin–Lam–T.–Williams '22)

For any *Coxeter word* $\vec{c}$, and $p > 0$ coprime to h,

$$\frac{1}{q^{rp/2}(1-q)^r} \sum_{\vec{w} \nearrow \vec{c}^p} q^{d_{\vec{w}}} (q-1)^{e_{\vec{w}}}$$

$$= \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{Deg}_\chi(\zeta_h)(1, [\Delta_{p/h}(\chi)]_q)_W$$

$$\Big(= (1, [L_{p/h}(1)]_q)_W \Big) \quad \leftarrow \quad q\text{-Catalan}$$

where:

- $\mathrm{Deg}_\chi = \mathrm{Deg}_\rho$ when $\chi_A^G(\rho) = \chi$.
- Δ and L denote Verma and simple rational DAHA modules.
- $[M]_q = \sum_i \mathrm{gr}_i\, q^{i/2} M_i \in K_0(W)(\!(q)\!)$.

Δ's are easy but infinite-dimensional.

L's are finite-dimensional but mysterious. The step in parentheses is:

(Berest–Etingof–Ginzburg, Chmutova–Etingof '03)

$$\sum_k (-1)^k [\Delta_{p/h}(\Lambda^k(V))]_q = [L_{p/h}(1)]_q.$$

Recall that $\mathrm{Deg}_\chi(\zeta_h) = \begin{cases} (-1)^k & \chi_A^G(\rho) = \Lambda^k(V), \\ 0 & \text{else.} \end{cases}$

(Trinh '21) Braid varieties of slope d/m described by

$$\sum_{\chi \in \mathrm{Irr}(W)} \mathrm{Deg}_\chi(\zeta_m)[\Delta_{d/m}(\chi)]_q.$$

3 Correspondences

Let $\hat{G}$ be the associated group over $\mathbf{C}(\!(z)\!)$. Again,

$$\{\text{maximal tori of } \hat{G}\}/\text{conj.} \quad \simeq \quad W/\text{conj.}$$

$\gamma \in \mathrm{Lie}\,\hat{G}$ is *homogeneous* of slope d/m iff

$$\gamma(c^m z) \underset{\text{conj.}}{\sim} c^d \gamma(z) \quad \text{for } c \in \mathbf{C}^\times.$$

For such γ generic* enough, $C_{\hat{G}}(\gamma)$ is a maximal torus whose type is regular of order m.

Example Take $\hat{G} = \mathrm{SL}_2(\mathbf{C}(\!(z)\!))$:

$\begin{pmatrix} z^d & \\ & -z^d \end{pmatrix}$ has slope d; $\begin{pmatrix} & 1 \\ z^d & \end{pmatrix}$ has slope $d/2$.

* regular semisimple

Fix Iwahori $I \subseteq \hat{G}$, affine flag variety $\mathcal{F}l_{\hat{G}} = \hat{G}/I$.

For generic homogeneous γ of slope d/m, the

affine Springer fiber $\mathcal{F}l_\gamma = \{gI \mid \gamma \in \operatorname{Lie} gIg^{-1}\}$

has a $\mathbf{C}^\times$-action studied by Lusztig–Smelt, Fan, Sage, Sommers, Varagnolo–Vasserot, ...

(Oblomkov–Yun '16) Cherednik action* on

$$E_\gamma = \mathrm{gr}^{\mathsf{P}}_* \, \mathrm{H}^*_{\mathbf{C}^\times, loc}(\mathcal{F}l_\gamma)_{st}.$$

- $\mathrm{gr}^{\mathsf{P}}_*$ needed for rational, not trigonometric, DAHA.
- st needed to avoid endoscopic(!) DAHAs.

* assuming γ elliptic

Let $T \subseteq G$ be regular of order m, like $C_{\hat{G}}(\gamma)$.

Conjecture (T.–Xue '23) $\mathrm{End}_{\mathrm{RCA}}(E_\gamma) \simeq H^G_T(1)$.

$$[E_\gamma]_q = \sum_{\rho \in \mathrm{Uch}^G_A \cap \mathrm{Uch}^G_T} [\Delta_{d/m}(\chi^G_A(\rho))]_q \otimes \chi^G_T(\rho)|_{x \to 1}$$

on graded characters.

$\mathbf{F}_q$	(q, q)	$\mathrm{H}^*_c(X_e) \otimes_G \mathrm{H}^*_c(X_w)$
$\mathbf{C}(\!(z)\!)$	$(\zeta_m, 1)$	E_γ

Theorem (T.–Xue '23) The conjecture holds when:

- m is the (twisted) Coxeter number of $\hat{G}$.
- $(\hat{G}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.

Led Ting and I to study, more generally,

$$\mathrm{Irr}(W^G_{L,\lambda}) \xleftarrow{\chi^G_{L,\lambda}} \mathrm{Uch}^G_{L,\lambda} \cap \mathrm{Uch}^G_{M,\mu} \xrightarrow{\chi^G_{M,\mu}} \mathrm{Irr}(W^G_{M,\mu})$$

for Φ_l-*cuspidal* (L,λ) and Φ_m-*cuspidal* (M,μ).

Example $(L,\lambda) = (A,1)$ and $(M,\mu) = (T,1)$ give

$$W^G_{A,1} = W \quad \text{and} \quad W^G_{T,1} = C_W(w).$$

Level–Rank Conjecture (T.–Xue '23)

1. $\mathrm{im}(\chi^G_{L,\lambda})$ describes a union of $H^G_{L,\lambda}(\zeta_m)$-blocks.
   $\mathrm{im}(\chi^G_{M,\mu})$ describes a union of $H^G_{M,\mu}(\zeta_l)$-blocks.
2. The diagram gives a *bijective matching* of blocks.

(T.–Xue '23) True for $G = \mathrm{GL}_n$ and l, m coprime.

Proved by embedding the diagram within the abacus combinatorics of Uglov.

It becomes a form of *level-rank duality*.

Abstract to appear in *Math. Res. Lett.* (2026). arXiv:2508.07051

Full (v1 2023, v2 2025). arXiv:2311.17106

Thank you for listening.