Braids, Unipotent Representations, and Nonabelian Hodge Theory

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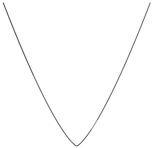
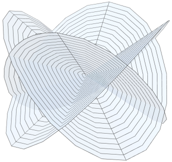
The curve below is singular at the origin:

$$y^3 = x^4.$$

Its real solutions *vs.* its complex solutions:





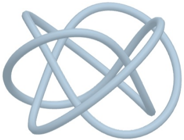
On the right, the self-intersections only exist in the projection to $\mathbf{R}^3$.

For small $\varepsilon > 0$, the preimage of the circle $|x| = \varepsilon$ is actually a *braided circle* in the curve.

In general, a plane curve singularity $a(x, y)$ gives rise to a knot or link. For

$$y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y + a_n(x),$$

the link is the closure of a braid on n strands.



The curve $y^3 = x^4$ gives rise to the torus knot $T_{3,4}$.

How do invariants of the link relate to invariants of a?

The local germ $C_a = \operatorname{Spec} \mathbf{C}[\![x, y]\!]/a(x, y)$ bears analogies with a global curve.

The scheme

$$\mathcal{J}_a = \left\{ \begin{array}{c} (C_a - 0)\text{-framed, torsion-free coherent} \\ \text{sheaves of degree 0 and generic rank 1} \end{array} \right\}$$

is the analogue of a Jacobian.

If a is irreducible, then $\mathcal{J}_a$ is a projective variety.

Conj (Oblomkov–Rasmussen–Shende)

If a is irreducible, then the homology of $\mathcal{J}_a$ embeds in the *HOMFLYPT homology* of the link of C_a.

Ex If $a(x, y) = y^3 - x^4$, then

$$[\mathcal{J}_a] = [pt] + [\mathbf{C}] + [\mathbf{C}^2] + [\mathbf{C}^2] + [\mathbf{C}^3].$$

In general, if $a(x, y) = y^n - x^m$ with m, n coprime, then $\mathcal{J}_a$ is paved by affine cells.

Pf sketch $\mathbf{C}[\![x, y]\!]/a(x, y) \simeq \mathbf{C}[\![t^m, t^n]\!]$.

Points of $\mathcal{J}_a$ correspond to $\mathbf{C}[\![t^m, t^n]\!]$-submodules $M \subseteq \mathbf{C}[\![t]\!]$ that satisfy

$$\begin{aligned} \operatorname{codim}(M) &= \operatorname{codim}(\mathbf{C}[\![t^m, t^n]\!]) && \text{(degree 0)}, \\ M \otimes \mathbf{C}(\!(t)\!) &\simeq \mathbf{C}(\!(t)\!) && \text{(rank 1)}. \end{aligned}$$

Rotating t induces $\mathbf{C}^\times \curvearrowright \mathcal{J}_a$. The fixed points are *monomial* modules; their attracting loci are cells.

Khovanov–Rozansky's HOMFLYPT homology:

$$\mathrm{HHH} : \{\text{links in } \mathbf{R}^3\}/\text{isotopy} \to \mathbf{Vect}_3$$

Let $\mathbf{P}(-) = \sum_{i,j,k} a^i q^{\frac{j}{2}} t^k \dim \mathrm{HHH}^{i,j,k}(-)$.

Ex For the $(3,4)$ torus knot,

$$\begin{aligned}\mathbf{P}(T_{3,4}) = a^6q^{-3}(&\mathbf{1} + q^2\mathbf{t^2} + q^3\mathbf{t^4} + q^4\mathbf{t^4} + q^6\mathbf{t^6}) \\ &+ a^8q^{-2}(t^3 + qt^5 + q^2t^5 + q^3t^7 + q^4t^7) \\ &+ a^{10}t^8.\end{aligned}$$

In general, $\mathrm{H}_*(\mathcal{J}_a)$ should be the "lowest a-degree" of $\mathrm{HHH}(\text{link of } C_a)$.

The q-variable tracks a *perverse filtration* that we'll discuss later.

Both sides can be interpreted in terms of $G = \mathrm{SL}_n$.

Let $F = \mathbf{C}(\!(x)\!)$ and $\mathcal{O} = \mathbf{C}[\![x]\!]$.

Recall the *affine Grassmannian* $\mathcal{G}r = G(F)/G(\mathcal{O})$.

If $\gamma \in \mathfrak{g}(\mathcal{O}) = \mathfrak{sl}_n(\mathcal{O})$ has characteristic polynomial

$$a(x,y) = \det(yI - \gamma),$$

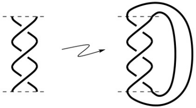
then the *affine Springer fiber*

$$\mathcal{G}r_\gamma = \{[g] \in \mathcal{G}r \mid \gamma \in \mathrm{Ad}(g) \cdot \mathfrak{g}(\mathcal{O})\} \subseteq \mathcal{G}r$$

is isomorphic to $\mathcal{J}_a$ via the map $[g] \mapsto M = g(\mathcal{O}^n)$.

Interpreting HHH via $G = \mathrm{SL}_n$ is more involved.

First, every link can be written as a braid closure $\hat{\beta}$.



If $\beta \in Br_n$, then we can compute $\mathbf{P}(\hat{\beta})|_{t=-1}$ from the value of β under

$$Br_n \to H_n \xrightarrow{\mathrm{tr}} \mathbf{Z}(q^{\frac{1}{2}})[a^{\pm 1}],$$

where H_n is the *Iwahori–Hecke algebra* of S_n and tr is a certain $q^{\frac{1}{2}}$-linear trace.

To get $\mathbf{P}(\hat{\beta})$ itself, we must categorify H_n and tr.

Let $\mathcal{B}$ be the flag variety.

Each G-orbit of $\mathcal{B} \times \mathcal{B}$ gives rise to a (pure) perverse sheaf called its intersection complex.

Their shift-twists generate an additive subcategory

$$\mathrm{C}(\mathcal{B} \times \mathcal{B}) \subseteq \mathrm{D}^{b,const}_{G.m}(\mathcal{B} \times \mathcal{B}).$$

Under a certain monoidal product, the *Hecke category*

$$\mathbf{H} = \mathrm{K}^b(\mathrm{C}(\mathcal{B} \times \mathcal{B}))$$

categorifies H_n. We can compute $\mathbf{P}(\hat{\beta})$ from

$$Br_n \xrightarrow{\mathcal{R}} \mathbf{H} \xrightarrow{\mathbf{Tr}} \mathbf{Vect}_3,$$

where $\mathcal{R}$ is due to Rouquier and $\mathbf{Tr}$ is a monoidal trace functor.

Let $\mathfrak{t}$ be the Cartan of $\mathfrak{g}$. We'll introduce a trace

$$\widetilde{\mathbf{Tr}} : \mathbf{H} \to \mathbf{Mod}_2(\mathbf{C}[W] \ltimes \mathrm{Sym}^*(\mathfrak{t})).$$

Thm 1 $\mathbf{Tr} \simeq \mathrm{Hom}_{S_n}(\Lambda^*(\mathfrak{t}), \widetilde{\mathbf{Tr}})$.

Let $\mathcal{U} \subseteq G$ be the unipotent locus. To each *positive* β, we'll assign a G-equivariant map

$$\mathcal{U}(\beta) \to \mathcal{U}.$$

For $\beta = \mathbf{1}$, it's the Springer resolution.

Thm 2 $\widetilde{\mathbf{Tr}}(\mathcal{R}(\beta)) \simeq \mathrm{gr}_*^{\mathrm{W}}\, \mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta))$.

By Springer theory, we deduce that $\mathrm{gr}_*^{\mathrm{W}}\, \mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta))$ is the lowest a-degree of $\mathrm{HHH}(\hat{\beta})$.

Suppose that:

- $\hat{\beta}$ is the link of (generically separable) $a(x, y)$.
- a is the characteristic polynomial of $\gamma \in \mathfrak{sl}_n(\mathbf{C}[\![x]\!])$.

Prop The following are equivalent:

- a is irreducible.
- $\mathcal{G}r_\gamma$ is a projective variety.
- $[\mathcal{U}(\beta)/G]$ has finite stabilizers.

Conj 1 Under the hypotheses above,

$$\mathcal{G}r_\gamma \xleftarrow{\text{deformation retract}} [\mathcal{U}(\beta)/G].$$

Moreover the halved weight filtration on $\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta))$ should match a "perverse" filtration on $\mathrm{H}_*(\mathcal{G}r_\gamma)$.

Motivated by a 2013 unpublished research statement of Shende's.

- B.-C. Ngô noticed that $\mathcal{G}r_\gamma$ resembles the fibers of *Hitchin's integrable system.*
- Boalch and Shende–Treumann–Williams–Zaslow related varieties similar to $\mathcal{U}(\beta)$ to *wild character varieties*, for special choices of β.

Nonabelian Hodge theory is about diffeomorphisms

$$\mathcal{M}_{Hitchin} \quad \sim \quad \mathcal{M}_{deRham} \quad \sim \quad \mathcal{M}_{Betti}.$$

The dCHM "P = W" conjecture is roughly:

$$\begin{array}{c} \textit{perverse} \text{ filtrations} \\ \text{on } \mathcal{M}_{Hitchin} \end{array} \quad \simeq \quad \begin{array}{c} \text{halved } \textit{weight} \text{ filtrations} \\ \text{on } \mathcal{M}_{Betti} \end{array}$$

Let's describe $\mathcal{U}(\beta)$.

Let $w \mapsto \sigma_w$ be the canonical section to $Br_n \to S_n$.
Let $O_w \subseteq \mathcal{B} \times \mathcal{B}$ be the G-orbit indexed by $w \in S_n$.

Deligne showed that if $\beta = \sigma_{w_1} \cdots \sigma_{w_k}$, then

$$O(\beta) = O_{w_1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} O_{w_k}$$

only depends on β up to strict isomorphisms.

Form Cartesian squares:

$$\begin{array}{ccccc} O(\beta) & \longleftarrow & G(\beta) & \longleftarrow & \mathcal{U}(\beta) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} \times \mathcal{B} & \xleftarrow{act} & G \times \mathcal{B} & \xleftarrow{\iota} & \mathcal{U} \times \mathcal{B} \end{array}$$

Next, let's describe $\widetilde{\mathbf{Tr}}$.

The cocenter $H_n \to H_n/[H_n, H_n]$ is geometrized by pulling and pushing through

$$\mathcal{B} \times \mathcal{B} \xleftarrow{act} G \times \mathcal{B} \xrightarrow{pr} G.$$

Let $\mathbf{Ch} : \mathrm{D}^b_{G,m}(\mathcal{B}^2) \to \mathrm{D}^b_{G,m}(G)$ be

$$\mathbf{Ch} = \bigoplus_i {}^p\mathcal{H}^i \circ pr_* act^! [-i].$$

The essential images of $\mathrm{C}(\mathcal{B}^2)$ under $\mathbf{Ch}$ and $\iota^* \circ \mathbf{Ch}$ generate subcategories

$$\mathrm{C}(G) \subseteq \mathrm{D}^b_{G,m}(G), \quad \mathrm{C}(\mathcal{U}) \subseteq \mathrm{D}^b_{G,m}(\mathcal{U})$$

of shifted pure perverse sheaves.

In particular, $\mathrm{C}(\mathcal{U})$ contains the Springer sheaf $\mathcal{S}$.

Using *weight realization functors*

$$\rho : \mathrm{K}^b(\mathrm{C}(-)) \to \mathrm{D}^b_{G,m}(-),$$

we can build

$$\begin{array}{ccccc} \mathrm{K}^b(\mathrm{C}(\mathcal{B}^2)) & \xrightarrow{\mathbf{Ch}} & \mathrm{K}^b(\mathrm{C}(G)) & \xrightarrow{\iota^*} & \mathrm{K}^b(\mathrm{C}(\mathcal{U})) \\ \downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \rho} \\ \mathrm{D}^b_{G,m}(\mathcal{B}^2) & \xrightarrow{\mathbf{Ch}} & \mathrm{D}^b_{G,m}(G) & \xrightarrow{\iota^*} & \mathrm{D}^b_{G,m}(\mathcal{U}) \end{array}$$

Up to shifts, $\widetilde{\mathbf{Tr}}$ is the composition

$$\mathrm{K}^b(\mathrm{C}(\mathcal{B}^2)) \to \mathrm{D}^b_{G,m}(\mathcal{U}) \to \mathbf{Mod}_2(\mathbf{C}[W] \ltimes \mathrm{Sym}),$$

where the second arrow is $\mathrm{gr}^{\mathrm{W}}_* \operatorname{Ext}^*(-, \mathcal{S})$.

Proving $\mathbf{Tr} \simeq \mathrm{Hom}_{S_n}(\Lambda^*(\mathfrak{t}), \widetilde{\mathbf{Tr}})$ amounts to relating this setup over $\mathcal{U}$ with Webster–Williamson's over G. (Their use of weights is *qualitatively* different.)

Proving $\widetilde{\mathbf{Tr}}(\mathcal{R}(\beta)) \simeq \mathrm{gr}^{\mathrm{W}}_* \mathrm{H}^{\mathrm{BM},G}_*(\mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta))$ relies on intertwining K^b degree with weight degree. Note that $\rho(\mathcal{R}(\beta)) = (O(\beta) \to \mathcal{B}^2)_! \mathbf{C}$.

All this generalizes to any connected semisimple G.

Even the conjecture about $\mathcal{G}r_\gamma$ generalizes! Instead of charpolys and links, use

$$\gamma \mapsto a : \mathfrak{g}(\mathcal{O}) \to (\mathfrak{t} /\!\!/ W)(\mathcal{O})$$

and the generalized braid group $Br_W = \pi_1(\mathfrak{t}^{\mathrm{reg}} /\!\!/ W)$.

Finally, we consider the decategorification of $\widetilde{\mathbf{Tr}}$.

Let $[\beta]_q = \widetilde{\mathbf{Tr}}(\mathcal{R}(\beta))|_{t=-1}$, abusing notation.

Let $\mathcal{B}_u \subseteq \mathcal{B}$ be the Springer fiber above $u \in \mathcal{U}$.

Thm 3 Specializing q to a prime power,

$$[\beta]_q = \pm \frac{1}{|G(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{U}(\beta)_u(\mathbf{F}_q)| [\mathcal{B}_u]_q$$

in $\mathbf{Rep}(W)$, where $[\mathcal{B}_u]_q = \sum_i q^i \mathrm{H}^{2i}(\mathcal{B}_u)$.

Not just a corollary of **Thm 2**.

The virtual weight series of $[X/G]$ need not be the quotient of that of X by that of G.

Instead, the proof uses a strange formula

$$[\beta]_q = q^{|\beta|/2}\varepsilon \cdot \sum_i q^i \operatorname{Sym}^i(\mathfrak{t}) \cdot \sum_{\phi,\psi\in \operatorname{Irr}(W)} \{\phi,\psi\}\phi_q(\beta)\psi,$$

where $\{-,-\} : \operatorname{Irr}(W) \times \operatorname{Irr}(W) \to \mathbf{Q}$ is Lusztig's "exotic Fourier transform."

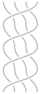
Ex Writing $r = \operatorname{rk}(W)$, we have

$$[\mathbf{1}]_q = \frac{1}{(1-q)^r}\,\mathbf{C}[W].$$

For $W = S_n$, recovers:

$$\mathbf{P}(n\text{-unlink})_{t=-1} = \left(\frac{a-a^{-1}}{q^{1/2}-q^{-1/2}}\right)^{n-1}.$$

The *full twist* is a central element $\pi = \sigma_{w_0}^2 \in Br_W^+$:



A braid β is *periodic of slope* $\frac{m}{n}$ iff $\beta^n = \pi^m$.

Thm 4 If β is periodic of slope ν, then

$$[\beta]_q = \sum_{\phi\in\operatorname{Irr}(W)} q^{\nu\mathbf{c}(\phi)}\operatorname{Deg}_\phi(e^{2\pi i\nu})\phi \cdot \sum_i q^i \operatorname{Sym}^i(\mathfrak{t}),$$

where:

- $\operatorname{Deg}_\phi(q)$ is the degree of the *unipotent principal series* of $G(\mathbf{F}_q)$ attached to ϕ.
- $\mathbf{c}(\phi)$ is the *content* of ϕ. For $W = S_n$, the content of the corresponding partition.

The key to **Thm 4**: π acts on irreps by scalars, and the traces $\phi_q(\beta)$ can be bootstrapped from $\phi_q(\pi)$.

Goes back to Jones's calculation of HOMFLYPT for torus knots.

Cor For W irreducible and n *cuspidal*,

$$[\beta]_q = \begin{cases} [L_\nu(1)]_q + [L_\nu(\mathrm{t})]_q & (W, n) = (E_8, 15) \text{ or } (H_4, 15) \\ [L_\nu(1)]_q & \text{else} \end{cases}$$

where $[L_\nu(\phi)]_q$ is the graded W-character of the simple *rational Cherednik module* indexed by ϕ.

Combined with work of Oblomkov–Yun, this is evidence for the $\mathcal{G}r_\gamma$ conjecture.

Thank you for listening.