



Haiman's Conjecture and the Springer Correspondence

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https://mqtrinh.github.io/math/research/haiman_26-04.pdf

1 Combinatorics Fix a graph with vertex set Γ .

A (*proper*) *coloring* of Γ is a function

$$\mathbf{c}: \Gamma \rightarrow \mathbf{Z}_+ \quad \text{s.t.} \quad \mathbf{c}(i) \neq \mathbf{c}(j) \text{ for all edges } i-j.$$

(Birkhoff 1912) The function

$$\chi_{\Gamma}(m) = |\{\mathbf{c} \mid \mathbf{c}(\Gamma) \subseteq \{1, 2, \dots, m\}\}|$$

is a polynomial in m .

(Stanley 1995) The *chromatic symmetric function*

$$X_{\Gamma}[x_1, x_2, \dots] = \sum_{\mathbf{c}} x_{\mathbf{c}}, \quad \text{where } x_{\mathbf{c}} = \prod_{i \in \Gamma} x_{\mathbf{c}(i)},$$

also interpolates the values $\chi_{\Gamma}(m)$.

Henceforth, $\Gamma = \{1, 2, \dots, n\}$.

(Shareshian–Wachs 2016) A *descent* of \mathbf{c} is

an edge $i-j$ s.t. $i < j$ and $\mathbf{c}(i) > \mathbf{c}(j)$.

Let $\text{des}(\mathbf{c})$ count the descents of \mathbf{c} .

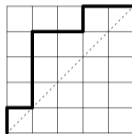
The *chromatic quasisymmetric function*

$$X_{\Gamma}(q)[x_1, x_2, \dots] = \sum_{\mathbf{c}} q^{\text{des}(\mathbf{c})} x_{\mathbf{c}}$$

need not be a symmetric function.

Yet it is for certain graphs arising from Dyck paths, or equivalently, Hessenberg sequences.

A 5×5 Dyck path:



Its Hessenberg sequence $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5)$:

$$(1, 4, 4, 5, 5).$$

Its *unit interval graph* $\Gamma(\mathbf{m})$:

$$i-j \text{ is an edge} \iff i < j \leq h(i).$$

Frobenius characteristic:

$$K_0(S_n) \xrightarrow{\sim} \{\text{degree-}n \text{ symmetric functions}\}$$

$$\chi^\lambda \mapsto \text{Schur } s_\lambda$$

$$\text{Ind}_{S_\lambda}^{S_n}(1) \mapsto \text{complete homogeneous } h_\lambda$$

$$\text{Ind}_{S_\lambda}^{S_n}(\text{sgn}) \mapsto \text{elementary } e_\lambda$$

Conj (Stanley–Stembridge 1993)

$$X_{\Gamma(\mathbf{m})}(1) \in \mathbf{Z}_{\geq 0}\langle e_\lambda \mid \lambda \vdash n \rangle.$$

Conj (Shareshian–Wachs 2016) Expand

$$X_{\Gamma(\mathbf{m})}(q) = \sum_{\lambda \vdash n} A_\lambda^{\mathbf{m}}(q) e_\lambda.$$

Then for each λ , the nonzero coefficients of $A_\lambda^{\mathbf{m}}(q)$ are positive and unimodal.

(Shareshian–Wachs 2016) Schur positivity holds
= positivity holds with s_λ in place of e_λ .

The coefficients even have a combinatorial meaning, in terms of inversions in Young tableaux.

Example If $\mathbf{m} = (1, 4, 4, 5, 5)$, then

$$\begin{aligned} X_{\Gamma(\mathbf{m})}(q) &= (1 + 3q + 4q^2 + 3q^3 + q^4) s_{111111} \\ &\quad + (1 + q)^4 s_{21111} \\ &\quad + (1 + q)^2 s_{2221} \\ &\quad + (1 + q)^2 s_{3111} \\ &= (1 + q)(1 + q + q^3) e_{41} \\ &\quad + q(1 + q)^2 e_{311}. \end{aligned}$$

(Brosnan–Chow 2017) Schur unimodality holds.

What Brosnan–Chow actually proved was a geometric interpretation of $X_{\Gamma(\mathbf{m})}(q)$.

Let $G = \mathrm{GL}_n$, so that $\mathfrak{g} = \mathfrak{gl}_n$.

Let B be upper-triangular, so that $\mathfrak{b} = \bigoplus_{i \leq j} \mathfrak{g}_{i,j}$.

Fix $g \in G(\mathbf{C})$. Using the *Hessenberg space*

$$\mathfrak{H}_{\mathbf{m}} = \mathfrak{b} \oplus \bigoplus_{i < j \leq \mathbf{m}_i} \mathfrak{g}_{j,i},$$

form the *Hessenberg variety*

$$\mathrm{Hess}_{\mathbf{m},g} = \{xB \in G/B \mid x^{-1}gx \in \mathfrak{H}_{\mathbf{m}}\}.$$

For regular semisimple g , the action of $\pi_1(G^{rs}, g)$ on its cohomology factors through the Weyl group S_n .

Let ω be the involution $h_\lambda \leftrightarrow e_\lambda$.

(Brosnan–Chow 2017) For any $g \in G^{rs}(\mathbf{C})$,

$$\omega X_{\Gamma(\mathbf{m})}(q) = \sum_i q^i \mathrm{FrobChar} \left[H^i(\mathrm{Hess}_{\mathbf{m},g}) \right].$$

Schur unimodality (and palindromicity) now follow from hard Lefschetz.

More recently, by pure algebra:

(Hikita, Griffin–Mellit–Romero–Weigl–Wen 2025)

$$X_{\Gamma(\mathbf{m})}(1) \in \mathbf{Z}_{\geq 0} \langle e_\lambda \mid \lambda \vdash n \rangle$$

= the Stanley–Stembridge conjecture is true.

Methods do not extend to $X_{\Gamma(\mathbf{m})}(q)$.

2 Characters

The e -positivity and -unimodality of $X_{\Gamma(\mathbf{m})}(q)$ are subsumed by a more general conjecture.

Let H_n be the Hecke algebra of S_n over $\mathbf{Z}[v^{\pm 1}]$.

Let $\{C'_w\}_w \subseteq H_n$ be Kazhdan–Lusztig's basis.

$$\chi^\lambda: S_n \rightarrow \mathbf{Z} \quad \text{deforms to} \quad \chi_v^\lambda: H_n \rightarrow \mathbf{Z}[v^{\pm 1}].$$

The nonzero coefficients of $\chi_v^\lambda(C'_w)$ are palindromic.

They are also positive and unimodal due to the irreducibility of left cells in S_n .

Conj (Haiman 1993) For any $z \in S_n$, expand

$$\sum_{\lambda \vdash n} \chi_v^\lambda(C'_z) s_\lambda = \sum_{\mu \vdash n} \alpha_\mu^z(\mathbf{v}) h_\mu.$$

Then for each μ , the nonzero coefficients of $\alpha_\mu^z(\mathbf{v})$ are positive and unimodal.

Haiman checked this up through $n = 7$.

Let $s_i = (i, i + 1)$ in S_n .

(Clearman–Hyatt–Shelton–Skandera 2016) For any \mathbf{m} ,

$$\omega X_{\Gamma(\mathbf{m})}(v^2) = v^{\ell(z)} \sum_{\lambda \vdash n} \chi_v^\lambda(C'_{z_{\mathbf{m}}}) s_\lambda$$

where $z_{\mathbf{m}} = (s_{\mathbf{m}_1-1} \cdots s_1) \cdots (s_{\mathbf{m}_{n-1}-1} \cdots s_{n-1})$.

Shareshian–Wachs

Haiman

\mathbf{m}

z

$$\omega X_{\Gamma(\mathbf{m})}(v^2)$$

$$\sum_{\lambda} \chi_v^{\lambda}(C'_z) s_{\lambda}$$

The $z_{\mathbf{m}}$'s are:

- smooth, in the sense of their Schubert varieties.
- precisely the *312-avoiding permutations*.
- counted by $\frac{(2n)!}{(n+1)!n!} \sim \frac{1}{\sqrt{\pi}} n^{-3/2} 4^n$.

Example If $\mathbf{m} = (1, 4, 4, 5, 5)$, then $z_{\mathbf{m}} = s_3 s_2 s_3 s_4$.

What is the meaning of $\text{Hess}_{\mathbf{m},g}$ and its cohomology in terms of $z_{\mathbf{m}}$?

Keeping $G = \text{GL}_n$, form the (*closed*) *Lusztig variety*

$$Y_{z,g} = \{xB \in G/B \mid x^{-1}gx \in \overline{BzB}\}.$$

Again, S_n acts on its cohomology.

(Abreu–Nigro 2024) For any $g \in G^{rs}(\mathbf{C})$ and \mathbf{m} ,

$$Y_{z_{\mathbf{m}},g} \simeq \text{Hess}_{\mathbf{m},g}.$$

The isomorphism on cohomology is S_n -equivariant.

Notice that Lusztig varieties exist for any reductive group G .

How much of the story generalizes beyond GL_n ?

Let G be connected reductive with Weyl group W .

Fix a Borel $B \subseteq G$ and element $z \in W$.

(Brosnan–J. Hong–D. Lee 2025) Fix $g \in G(\mathbf{C})$. Using

$$\mathfrak{H}_z = \mathfrak{b} \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ s_\alpha \leq z}} \mathfrak{g}_{-\alpha},$$

form $\text{Hess}_{z,g} = \{xB \in G/B \mid x^{-1}gx \in \mathfrak{H}_z\}$.

If z is smooth and g is regular semisimple, then:

- $\text{Hess}_{z,g}$ is smooth.
- There is a flat degeneration from $Y_{z,g}$ to $\text{Hess}_{z,g}$.
- The isomorphism on cohomology is W -equivariant.

3 Calculations

How to generalize Haiman's conjecture beyond GL_n ?

Haiman's conjecture is about the Laurent polynomials

$$\alpha_\mu^z(\mathbf{v}) \in \mathbf{Z}[\mathbf{v}^{\pm 1}]$$

defined by

$$\sum_{\chi \in \text{Irr}(S_n)} \chi_{\mathbf{v}}(C'_z) \chi = \sum_{\mu \vdash n} \alpha_\mu^z(\mathbf{v}) \text{Ind}_{S_\mu}^{S_n}(1).$$

Problem 1 Beyond S_n , even the positivity of $\chi_{\mathbf{v}}(C'_z)$ for irreducible χ can fail.

Problem 2 Beyond S_n , not enough conjugacy classes of parabolic subgroups to replace the S_μ .

Solution to (1) (T., Brosnan–Hong–Lee–E. Lee 2026)

Replace the left-hand side with the (intersection) cohomology of $Y_{z,g}$ for regular semisimple g .

Thm For any $g \in G^{rs}(\mathbf{C})$ and $w \in W$,

$$\sum_i v^{2i} \mathrm{IH}^{2i}(Y_{z,g}) = v^{\ell(z)} \sum_{\chi, \psi} \{\chi, \psi\} \psi_v(c_z) \chi$$

where $\{-, -\}$ is Lusztig's exotic Fourier transform truncated to $\mathrm{Irr}(W)$.

Mostly follows from (multiplicity formula of Lusztig) + (local constancy result of Abreu–Nigro).

Lets us compute tons of examples using CHEVIE.

Solution to (2) (T., Brosnan–Hong–Lee–E. Lee 2026)

Replace the $\mathrm{Ind}_{S_\mu}^{S_n}(1)$ with characters arising from the *Springer correspondence* for G .

Recall that this is an injective map

$$\mathrm{Irr}(W) \rightarrow \left\{ (u, \kappa) \left| \begin{array}{l} u \in G(\mathbf{C}) \text{ unipotent,} \\ \kappa \in \mathrm{Irr}(\pi_0(G_u)) \end{array} \right. \right\} / G.$$

If $\psi \in \mathrm{Irr}(W)$ is the character of $H^{\mathrm{top}}(\mathcal{B}_u)_\kappa$ under the Springer action, then $\psi \mapsto [u, \kappa]$.

In this case, set $\mathit{spr}_\psi = \sum_j (-1)^j H^j(\mathcal{B}_u)_\kappa$.

Example If $\psi = \chi^\mu \in \mathrm{Irr}(S_n)$, then $\mathit{spr}_\mu = \mathrm{Ind}_{S_\mu}^{S_n}(1)$.

So beyond type A , study the nonzero coefficients of the Laurent polynomials

$$\alpha_{\psi, G}^z(\mathbf{v}) \in \mathbf{Z}[\mathbf{v}^{\pm 1}]$$

defined by

$$\sum_{\chi, \psi} \{\chi, \psi\} \psi_{\mathbf{v}}(c_z) \chi = \sum_{\psi \in \text{Irr}(W)} \alpha_{\psi, G}^z(\mathbf{v}) \text{spr}_{\psi}$$

! Although positivity usually holds, there is already a (single) failure in type B_2 .

! Even up to sign, unimodality can fail in type C_4 .

Example The $\alpha_{\psi, G}^z$ in type G_2 .

	e	2	1	$21, 12$	212	121
$\phi_{1,0}$				(101)	(1020)	(1020)
$\phi'_{1,3}$						
$\phi_{2,1}$				-1	-(10)	-(10)
$\phi_{2,2}$		(10)		1		(10)
$\phi'_{1,3}$			(10)	1	(10)	
$\phi_{1,6}$	1					

	$2121, 1212$	21212	12121	w_o
$\phi_{1,0}$	(10202)	(102020)	(102020)	(1020202)
$\phi'_{1,3}$	1	(10)	(10)	
$\phi_{2,1}$	-1			
$\phi_{2,2}$			-(10)	
$\phi'_{1,3}$	1		(10)	
$\phi_{1,6}$				

Above, $(a_m \cdots a_1 a_0) := a_0 + \sum_{i=1}^m a_i (\mathbf{v}^i + \mathbf{v}^{-i})$.

What properties of z or ψ ensure that $\alpha_{\psi, G}^z$ has positive unimodal coefficients?

Inclusion Conj (T. 2026)

Suppose that $z \in W'$ for some parabolic $W' \subseteq W$, corresponding to a Levi $G' \subseteq G$.

If $\alpha_{\psi', G'}^z$ is positive and unimodal for all $\psi' \in \text{Irr}(W')$, then so is $\alpha_{\psi, G}^z$ for all $\psi \in \text{Irr}(W)$.

Thm (T. 2026)

True whenever W' is a product of symmetric groups:

The $\alpha_{\psi, G}^z$ are $\mathbf{Z}_{\geq 0}$ -linear combinations of the $\alpha_{\psi', G'}^z$.

Proof

$\text{IH}^{2i}(Y_{z,g})$ and $\sum_j (-1)^j H^j(\mathcal{B}_u)$ enjoy the same compatibility with induction from W' to W .

Inflation Conj (T. 2026)

Suppose that ψ is inflated from a product of symmetric groups.

Then for any $z \in W$, the nonzero coefficients of $\alpha_{\psi, G}^z$ all have the same sign and are unimodal mod sign.

If the inflation morphism does not descend to

$$W(G_2) \rightarrow W(A_2) = S_3,$$

then the sign is positive.

This conjecture holds up through rank 5, for rationally smooth z in rank 6, and for many rank 7 examples.

Example Type F_4 .

		unimodal	positive	inflated
$\phi_{1,0}$	F_4			✓
$\phi'_{2,4}$	$F_4(a_1)(11)$			✓
$\phi_{4,1}$	$F_4(a_1)$	×	×	
$\phi''_{2,4}$	$F_4(a_2)(11)$			✓
$\phi_{9,2}$	$F_4(a_2)$	×	×	
$\phi'_{8,3}$	C_3			
$\phi''_{8,3}$	B_3			
$\phi'_{1,12}$	$F_4(a_3)(211)$			✓
$\phi''_{6,6}$	$F_4(a_3)(22)$			
$\phi'_{9,6}$	$F_4(a_3)(31)$		×	
$\phi_{12,4}$	$F_4(a_3)$		×	
$\phi_{4,7}$	$C_3(a_1)(11)$		×	
$\phi_{16,5}$	$C_3(a_1)$		×	
$\phi'_{6,6}$	$\tilde{A}_2 + A_1$		×	
$\phi_{4,8}$	$B_2(11)$			✓
$\phi''_{9,6}$	B_2		×	
$\phi''_{4,7}$	$A_2 + \tilde{A}_1$		×	
$\phi'_{8,9}$	\tilde{A}_2			
$\phi''_{1,12}$	$A_2(11)$			✓
$\phi''_{8,9}$	A_2			
$\phi_{9,10}$	$A_1 + \tilde{A}_1$			
$\phi'_{2,16}$	$\tilde{A}_1(11)$			✓
$\phi_{4,13}$	\tilde{A}_1		×	
$\phi''_{2,16}$	A_1			✓
$\phi_{1,24}$	1			✓

Webpage with data and accompanying code: <https://mqtrinh.github.io/math/research/code/chevie/>

Claude Sonnet 4.6 helped me write the regex scripts.

Thank you for listening.