Catalan Combinatorics in Algebraic Geometry

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Plan of talk:

0. Catalan Numbers
1. Dyck Paths
2. Braids
3. Deograms
4. Coda: NAHT

References for §2–3:

- [T] arXiv:2106.07444
- [Galashin–Lam–T–Williams] arXiv:2208.00121

§0 Catalan Numbers

The Catalan numbers generalize in several ways: rational slopes, Coxeter groups, q-numbers...

As we generalize them, we encounter two paradigms for the collections of objects they enumerate:

nonnesting *versus* noncrossing

nonnesting generalize to Weyl groups (Postnikov), admit Dyck-path-like statistics

noncrossing generalize to Coxeter groups (Reiner, Bessis), depend on a Coxeter element

The tension between these has interesting incarnations in algebraic geometry.

$$\mathfrak{C}_n := \frac{(2n)!}{(n+1)!n!}$$

$$\mathfrak{C}_{d/n} := \frac{(d+n-1)!}{d!n!} \qquad \text{for coprime } d, n$$

$$(\mathfrak{C}_{1/3}, \mathfrak{C}_{2/3}, \mathfrak{C}_{4/3}, \mathfrak{C}_{5/3}, \ldots) = (1, 2, 5, 7, \ldots)$$

$$[n] := 1 + q + \cdots + q^{n-1}$$

$$[n]! := [1][2]\cdots[n]$$

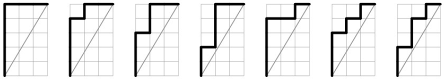
$$\mathfrak{C}_{d/n}(q) = \frac{[d+n-1]!}{[d]![n]!} \qquad \text{for coprime } d, n$$

$$\mathfrak{C}_{4/3}(q) = 1 + q^2 + q^3 + q^4 + q^6$$

Lastly, a bivariate $\mathfrak{C}_{d/n}(q,t)$, more difficult to define.

§1 Dyck Paths

$\mathfrak{C}_{d/n}$ counts the lattice paths above the diagonal in a $d \times n$ rectangle:



Above are the Dyck paths of slope $\frac{5}{3}$.

Piontkowski gave a variety stratified by $\mathfrak{C}_{d/n}$-many affine spaces of various dimensions.

Gorsky–Mazin matched the strata with Dyck paths.

Hikita interpreted $\mathfrak{C}_{d/n}(q,t)$ in this geometry.

We'll explain the construction in Lie-theoretic terms.

Let $F = \mathbf{C}(\!(x)\!)$ and $\mathcal{O} = \mathbf{C}[\![x]\!]$.

Let $G = \mathrm{SL}_n$. The affine Grassmannian of G is

$$\mathcal{G}r_n = G(F)/G(\mathcal{O}).$$

It has a Cartan decomposition

$$\mathcal{G}r_n = \coprod_{\mu \in X_+^\vee} \underbrace{G(\mathcal{O})x^\mu G(\mathcal{O})/G(\mathcal{O})}_{\mathcal{G}r_\mu},$$

where $X_+^\vee = \{\mu \in \mathbf{Z}^n \mid \mu_i \text{ decreasing and zero-sum}\}$,

$$x^\mu = \begin{pmatrix} x^{\mu_1} & & \\ & \ddots & \\ & & x^{\mu_n} \end{pmatrix}.$$

Any $\nu \in X_+^\vee$ defines an action $\mathbf{C}^\times \curvearrowright G(F)$:

$$c \cdot_\nu g(x) = c^\nu g(c^{2n}x)c^{-\nu}.$$

Induces an action $\mathbf{C}^\times \curvearrowright \mathcal{G}r_n$.

Generic fixed points are cosets $[x^{w\mu}]$ for $w \in S_n$.

Also induces an action $\mathbf{C}^\times \curvearrowright \mathfrak{g}(F) = \mathfrak{sl}_n(F)$.

Lem If $\gamma \in \mathfrak{g}(F)$ is an eigenvector of $\mathbf{C}^\times$, then

$$\mathcal{G}r_n(\gamma) = \{[g] \in \mathcal{G}r_n \mid g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O})\}$$

is stable under the ν-action on $\mathcal{G}r_n$.

We'll pick ν and γ so that ν-fixed points of $\mathcal{G}r_n(\gamma)$ correspond to Dyck paths of slope $\frac{d}{n}$.

Let $\{\alpha_i\}_i \subseteq \Phi^+ \subseteq \Phi$ be the simple roots.

$$\nu_d = d \begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & 1-n \end{pmatrix} = \sum_{\alpha \in \Phi^+} d\alpha^\vee,$$

$$\gamma_d = \begin{pmatrix} & & x^d \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = x^d e_{\alpha_{\mathrm{top}}} + \sum_i e_{-\alpha_i},$$

where $\mathfrak{g}_\alpha = \mathbf{C}e_\alpha$ and α_{top} is the highest root.

Lem (Lusztig–Smelt, Sommers) γ_d is an eigenvector of the ν_d-action on $\mathfrak{g}(F)$. Moreover,

$$\mathcal{G}r_n(\gamma_d)^{\nu_d} = \left\{ [x^\mu] \in \mathcal{G}r_n \;\middle|\; \begin{array}{l} \mu \in X_+^\vee, \\ \langle \alpha_{\mathrm{top}}, \mu \rangle \leq d \end{array} \right\}.$$

Let $\delta = \frac{1}{2}(d-1)(n-1)$, and let

$$J_{d/n} = \left\{ \Delta \subseteq \mathbf{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \Delta + d\mathbf{Z}_{\geq 0} + n\mathbf{Z}_{\geq 0} \subseteq \Delta, \\ |\mathbf{Z}_{\geq 0} \setminus \Delta| = \delta \end{array} \right\}.$$

Lem Explicit bijection $\mathcal{G}r_n(\gamma_d)^{\nu_d} \xrightarrow{\sim} J_{d/n}$:

$$[x^\mu] \mapsto \coprod_i (\underbrace{n\mu_i + d(i-1)}_{a_i(\mu)} + n\mathbf{Z}_{\geq 0}).$$

Ex Take $\frac{d}{n} = \frac{4}{3}$.

μ	$(a_1(\mu), a_2(\mu), a_3(\mu))$
$(0,0,0)$	$(0,4,8)$
$2\alpha_1^\vee + \alpha_2^\vee = (2,-1,-1)$	$(6,1,5)$
$\alpha_1^\vee + 2\alpha_2^\vee = (1,1,-2)$	$(3,7,2)$
$2\alpha_1^\vee + 2\alpha_2^\vee = (2,0,-2)$	$(6,4,2)$
$\alpha_1^\vee + \alpha_2^\vee = (1,0,-1)$	$(3,4,5)$

Lem Explicit bijection from $J_{d/n}$ to the set of Dyck paths of slope $\frac{d}{n}$.

Ex Let $\min(\mu) = \min_{1 \leq i \leq n-1} a_i(\mu)$.

If $\mu = (2, 0, -2)$, then $\min(\mu) = 2$ and

$$\coprod_i (a_i(\mu) - \min(\mu) + 3\mathbf{Z}_{\geq 0}) = \{0, 2, 3, 4, 5, \ldots\}$$

$\Longrightarrow$

$\vdots$	$\vdots$	$\vdots$
11	7	3
8	4	0
5	1	
2		

Cor $|\mathcal{G}r_n(\gamma_d)^{\nu_d}| = \mathfrak{C}_{d/n}$.

Lem The strata $\mathcal{G}r_\mu(\gamma_d) = \mathcal{G}r_\mu \cap \mathcal{G}r_n(\gamma_d)$ are affine spaces (when nonempty).

Thm (Gorsky–Mazin + Hikita)

$$\mathfrak{C}_{d/n}(q, t) = \sum_{[x^\mu] \in \mathcal{G}r_n(\gamma_d)^{\nu_d}} q^{\delta - \min(\mu)} t^{\dim(\mathcal{G}r_\mu(\gamma))}.$$

Both sides specialize to $\mathfrak{C}_{d/n}(q)$ when $q = t$.

Ex $\mathfrak{C}_{4/3}(q, t) = 1 + qt + qt^2 + q^2t^2 + q^3t^3$.

μ	$\delta - \min(\mu)$	$\dim(\mathcal{G}r_\mu(\gamma))$
$(0, 0, 0)$	3	3
$(2, -1, -1)$	2	2
$(1, 1, -2)$	1	2
$(2, 0, -2)$	1	1
$(1, 0, -1)$	0	0

Now let G be any almost-simple, simply-connected algebraic group.

We can replace $\mathcal{G}r_n$ with $\mathcal{G}r_G$, and replace n with the Coxeter number h of the Weyl group W.

Let $d_1, \ldots, d_r$ be the invariant degrees and

$$\mathfrak{C}_{W,d}(q) := \prod_{1 \leq i \leq r} \frac{[d + \overline{d(d_i - 1)}]}{[d_i]},$$

where $\overline{d(d_i - 1)}$ is the remainder of $d(d_i - 1)$ mod h.

Set $\mathfrak{C}_{W,d} := \mathfrak{C}_{W,d}(1)$.

Thm (Oblomkov–Yun) $|\mathcal{G}r_G(\gamma_d)^{\nu_d}| = \mathfrak{C}_{W,d}$.

Proof uses a cohomological rational Cherednik algebra.

But Hikita's combinatorics do not generalize.

Thm (T) For $G = \mathrm{Sp}_4$, no "reasonable" analogue of Hikita's construction recovers $\mathfrak{C}_{W,d}(q)$ from $\mathcal{G}r_G(\gamma_d)$.

Nonetheless, a construction of *noncrossing* rather than *nonnesting* flavor gives:

Thm (T) There is a G-variety $\mathcal{U}_{G,d}$ such that

$$\mathfrak{C}^{\mathrm{geo}}_{W,d}(q, t) := \sum_{j,k} q^{\frac{j}{2}} t^k \, \mathrm{gr}^{\mathsf{W}}_j \, \mathrm{H}^k_{c,G}(\mathcal{U}_{G,d})$$

satisfies:

1. $\mathfrak{C}^{\mathrm{geo}}_{W,d}(q, t) = \mathfrak{C}_{d/n}(q, qt^2)$ when $G = \mathrm{SL}_n$.
2. $\mathfrak{C}^{\mathrm{geo}}_{W,d}(q, -1) = |\mathcal{U}_{G,d}(\mathbf{F}_q)|/|G(\mathbf{F}_q)| = \mathfrak{C}_{W,d}(q)$.

Above, W is a so-called weight filtration.

§2 Braids

$\mathcal{U} \subseteq G$ unipotent variety, $\mathcal{B}$ flag variety

For $B, B' \in \mathcal{B}$, we write $B \xrightarrow{w} B'$ to mean $B' \subseteq B\dot{w}B$.

For any tuple of simple reflections $\vec{s} = (s_1, \ldots, s_\ell)$, let

$$\mathcal{U}(\vec{s}) = \{(u, \vec{B}) \in \mathcal{U} \times \mathcal{B}^\ell \mid B_\ell^u \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell\}$$

where $B^u = u^{-1}Bu$. Action of G by conjugation.

Lem If $\vec{s}$ changes by a braid move, then $\mathcal{U}(\vec{s})$ changes by a fixed isomorphism that preserves u.

Up to these isomorphisms, $\mathcal{U}(\vec{s})$ only depends on the underlying braid β of $\vec{s}$, so we write $\mathcal{U}(\beta)$.

G also acts on

$$\begin{aligned} \widetilde{\mathcal{U}}(\beta) &= \{(u, \vec{B}, B') \in \mathcal{U}(\beta) \times \mathcal{B} \mid u \in B'\}, \\ \mathcal{X}(\beta) &= \{(1, \vec{B}) \in \mathcal{U}(\beta)\} \\ &= \{\vec{B} \in \mathcal{B}^\ell \mid B_\ell \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell\}. \end{aligned}$$

The fibers of $\widetilde{\mathcal{U}}(\beta) \to \mathcal{U}(\beta)$ are *Springer fibers*, which have a W-action on cohomology.

Thm (T) There's a W-action on $\mathrm{H}^*_{c,G}(\widetilde{\mathcal{U}}(\beta))$ such that:

1. The invariants are $\mathrm{H}^*_{c,G}(\mathcal{U}(\beta))$.
2. The anti-invariants (sgn-isotypics) are $\mathrm{H}^*_{c,G}(\mathcal{X}(\beta))$.

(We actually need a derived version of $\tilde{\mathcal{U}}(\beta)$.)

The *superpolynomial* is an isotopy invariant

$$\mathbf{P} : \{\text{links in } \mathbf{R}^3\} \to \mathbf{Z}[\![q]\!][a^{\pm 1}, t^{\pm 1}]$$

$\mathbf{P}|_{t \to -1}$ is the HOMFLYPT series in a and $q^{\frac{1}{2}}$.

$\mathbf{P}$ is itself the graded dimension of HOMFLYPT or Khovanov–Rozansky homology.

For $V \in \mathrm{Rep}(S_n)$, let $V[\wedge^i] = \mathrm{Hom}_{S_n}(\bigwedge^i(\mathbf{C}^{n-1}), V)$.

Note that $[\wedge^0]$ = invariants, $[\wedge^{n-1}]$ = anti-invariants.

Thm (T) Take $G = \mathrm{SL}_n$, so that $W = S_n$.

If $\hat{\beta}$ is the link closure of β, then

$$\mathbf{P}(\hat{\beta}) \propto \sum_{i,j,k} (a^2 q^{\frac{1}{2}} t)^i q^{\frac{i-j}{2}} t^{k-j} \, \mathrm{gr}^{\mathsf{W}}_j \, \mathrm{H}^k_{c,G}(\widetilde{\mathcal{U}}(\beta))[\wedge^i].$$

Let β_d be a braid on n strands whose link closure is the (d, n) torus knot $T_{d,n}$.

Mellit, building on Elias–Hogancamp, computed $\mathbf{P}(T_{d,n})$. Lowest a-degree part is $q^{-\delta}\mathfrak{C}_{d/n}(q, qt^2)$.

Cor (T) $\mathrm{gr}^{\mathsf{W}}_* \, \mathrm{H}^*(\mathcal{U}(\beta_d))$ encodes $\mathfrak{C}_{d/n}(q, qt^2)$.

Nakagane, building on Kálmán, showed that

$$\text{lowest } a\text{-degree of } \mathbf{P}(T_{d,n}) \;\propto\; \text{highest } a\text{-degree of } \mathbf{P}(T_{d+n,n})$$

(GHMN generalized to the full twist of any braid.)

Cor (T) $\mathrm{gr}^{\mathsf{W}}_* \, \mathrm{H}^*(\mathcal{U}(\beta_d)) \simeq \mathrm{gr}^{\mathsf{W}}_* \, \mathrm{H}^*(\mathcal{X}(\beta_{d+n}))$.

Ex Take $\frac{d}{n} = \frac{3}{2}$, so that $\vec{s} = (s_1, s_1, s_1)$. The link closure $\hat{\beta}$ is a trefoil, for which

$$\mathbf{P}(\hat{\beta}) = a^2 q^{-1} + a^2 q t^2 + a^4 t^3.$$

Note that $[a^{-2} q\, \mathbf{P}(\hat{\beta})]|_{a \to 0} = \mathfrak{C}_{3/2}(q, qt^2)$.

Meanwhile, $\mathrm{gr}^{\mathsf{W}}_* \, \mathrm{H}^*_{c,G}(\widetilde{\mathcal{U}}(\beta_d))$ looks like:

	$\mathrm{gr}^{\mathsf{W}}_0$	$\mathrm{gr}^{\mathsf{W}}_2$	$\mathrm{gr}^{\mathsf{W}}_4$
H^2_c	$\bigwedge^0$		
H^4_c		$\bigwedge^1$	$\bigwedge^0$

where $\bigwedge^0 = \mathrm{triv}$, $\bigwedge^1 = \mathrm{sgn}$

The generating function for $\mathrm{gr}^{\mathsf{W}}_j \, \mathrm{H}^k_{c,G}[\wedge^i]$ is

$$q^{\frac{1}{2}} t^2 + (a^2 q^{\frac{1}{2}} t) q^{-1} t^2 + q^{-\frac{3}{2}}.$$

We can generalize β_d to any G, using the Coxeter number h in place of n.

Thm (T) The Armstrong–Reiner–Rhoades parking space of (W, d) is

$$\bigoplus_j \mathrm{gr}^{\mathsf{W}}_j \, \mathrm{H}^*_{c,G}(\widetilde{\mathcal{U}}(\beta_d)).$$

Cor (T) $|\mathcal{U}(\beta_d)(\mathbf{F}_q)/G(\mathbf{F}_q)| \propto \mathfrak{C}_{W,d}(q, -1)$ in all types.

Sommers defined a certain decomposition

$$C_{W,d}(q) = \sum_{[u] \in \mathcal{U}/G} Kr_{[u],d}(q),$$

recovering the usual Kreweras numbers for $W = S_n$.

Cor (T) If $\mathcal{U}(\beta_d, [u]) \subseteq \mathcal{U}(\beta_d)$ is the preimage of $[u]$, then $|\mathcal{U}(\beta_d, [u])(\mathbf{F}_q)/G(\mathbf{F}_q)| \propto Kr_{[u],d}(q)$.

§3 Deograms

We know much more about $\mathcal{X}(\beta)$ than $\mathcal{U}(\beta), \widetilde{\mathcal{U}}(\beta)$.

Fix $B_+ \in \mathcal{B}$ and $w \in W$. Let

$$X_+(\beta, w) = \left\{ \vec{B} \;\middle|\; \begin{array}{l} B_+ \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell, \\ B_\ell^w = B_+ \end{array} \right\}.$$

Lem $[\mathcal{X}(\beta\sigma_w)/G] \simeq [X_+(\beta, w)/(B_+ \cap B_+^w)]$.

Thm (CGGLSS, GLSB) $X_+(\beta, w)$ is a cluster variety.

For $u \in W$, the *open Richardson variety* of GLTW is

$$R^\circ_{u,\beta} := X_+(\beta\sigma_{w_0 u^{-1}}, w_0),$$

where $w_0 \in W$ is the longest element and σ_w is the braid lift of w.

A *u-Deogram* of $\vec{s} = (s_1, \ldots, s_\ell)$ is

$$\vec{x} = (x_1, \ldots, x_\ell) \quad \text{s.t.} \left\{ \begin{array}{l} x_i \in \{e, s_i\} \; \forall i, \\ x_1 \cdots x_i \leq x_1 \cdots x_{i-1} s_i \; \forall i, \\ u = x_1 \cdots x_\ell \end{array} \right.$$

Let $\mathcal{D}_u(\vec{s})$ be the set of all u-Deograms of $\vec{s}$. Let

$$\mathbf{e}_{\vec{x}} = \{i \mid x_i = e\}, \quad \mathbf{d}_{\vec{x}} = \{i \mid x_1 \cdots x_i < x_1 \cdots x_{i-1}\}.$$

Let $\mathcal{M}_u(\vec{s}) \subseteq \mathcal{D}_u(\vec{s})$ consist of $\vec{x}$ that minimize $|\mathbf{e}_{\vec{x}}|$.

Thm (Deodhar) If β arises from $\vec{s}$, then

$$R^\circ_{u,\beta} = \coprod_{\vec{x} \in \mathcal{D}_u(\vec{s})} ((\mathbf{C}^\times)^{\mathbf{e}_{\vec{x}}} \times \mathbf{C}^{\mathbf{d}_{\vec{x}}}).$$

Thm (GLTW) If $\beta = \beta_d$, then $|\mathcal{M}_e(\vec{s})| = \mathfrak{C}_{W,d}$.

In fact, $(q-1)^{-\mathrm{rk}(G)} |R^\circ_{e,\beta_d}(\mathbf{F}_q)| = \mathfrak{C}_{W,d}(q)$.

Lusztig's truncated F.T. on $\mathrm{Irr}(W)$ essential to proof.

§4 Coda

How is $\mathcal{U}_{G,d} := \mathcal{U}(\beta_d)$ related to $\mathcal{G}r_G(\gamma_d)$?

Bezrukavnikov–Boxeida–McBreen–Yun recently constructed a "wild Hitchin fibration"

$$f_{G,d} : \mathcal{M}_{G,d} \to \mathcal{A}_{G,d}$$

and an action $\mathbf{C}^\times \curvearrowright \mathcal{M}_{G,d}$, which contracts it onto a fiber of $f_{G,d}$ homeomorphic to $\mathcal{G}r_G(\gamma_d)$.

Writing-in-progress $[\mathcal{U}_{G,d}/G]$ and $\mathcal{M}_{G,d}$ are homeomorphic at the level of coarse spaces.

Arises from nonabelian Hodge theory on $\mathbf{CP}^1$ with:

- a regular singularity at $x = 0$ of nilpotent residue,
- a wild singularity of type $\gamma_d \frac{dx}{x}$ at $x = \infty$.

$\mathcal{N} \subseteq \mathfrak{g}$ nilpotent variety

There are spaces $\mathcal{F}l_G(\gamma_d)$ and $\widetilde{\mathcal{M}}_{G,d}$ that we expect to fit into a diagram:

$$\begin{array}{ccccc}
\mathcal{F}l_G(\gamma_d) & \xleftarrow{\textit{retract}} & \widetilde{\mathcal{M}}_{G,d} & \xrightarrow{\sim} & \widetilde{\mathcal{U}}_{G,d}/G \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G}r_G(\gamma_d) & \xleftarrow{\textit{retract}} & \mathcal{M}_{G,d} & \xrightarrow{\sim} & \mathcal{U}_{G,d}/G \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}/G & = & \mathcal{N}/G & = & \mathcal{U}/G
\end{array}$$

The first row is "parking". The second is "Catalan".

Thank you for listening.