The Topology and Arithmetic of Unipotent Braid Varieties

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I'll use $S_n = \langle s_1, \ldots, s_{n-1} \rangle$, $Br_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$.

My σ_i maps to the $q^{-1/2} T_i$ in

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1 Braids and Links

Let $\mathbf{Z}_{a,q} = \mathbf{Z}[a^{\pm 1}](q^{1/2})$.

The *reduced HOMFLYPT skein algebra* of a surface:

$$\mathrm{Sk}(\Sigma) = \mathbf{Z}_{a,q} \langle \text{oriented link diagrams in } \Sigma \rangle / \sim$$

where $\sim$ is generated by isotopy and

$$\bigcirc = 1,$$

$$a \; \diagup\!\!\diagdown \; - a^{-1} \; \diagdown\!\!\diagup \; = (q^{1/2} - q^{-1/2}) \;)\!(.$$

The ring structure is superposition of diagrams.

(HOMFLYPT 1985) $\mathrm{Sk}(\mathbf{R}^2) \simeq \mathbf{Z}_{a,q}$.

How about the punctured plane $\mathbf{R}^2 - \mathbf{0}$?

Pairwise distinct p_k for $k \in \mathbf{Z}$.

$p_5 =$ 

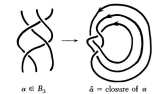
(Turaev 1988) $\mathrm{Sk}(\mathbf{R}^2 - \mathbf{0}) \simeq \mathbf{Z}_{a,q}[p_{\pm 1}, p_{\pm 2}, \ldots]$.

For $n \geq 0$, let

$$\mathrm{Sk}_n(\mathbf{R}^2 - \mathbf{0}) = \mathbf{Z}_{a,q}\langle p_\lambda \mid \lambda \vdash n \rangle$$

where $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$

Annular closure of braids



can be viewed as a composition

$$Br_n \xrightarrow{\mathsf{ann}} \mathrm{Sk}_n(\mathbf{R}^2 - \mathbf{0}) \xrightarrow{\mathsf{fill}} \mathrm{Sk}(\mathbf{R}^2).$$

We have $p_\lambda = \mathsf{ann}(\sigma_w)$, where w is of cycle type λ.

Problem How to categorify ann and fill?

We'll give an answer in terms of *braid varieties*.

Key Idea Representation theory of S_n.

$$\mathrm{Sk}_n(\mathbf{R}^2 - \mathbf{0}) \overset{*}{\simeq} \{ \text{ class functions } S_n \to \mathbf{Z}_{a,q} \}$$
$$= \mathbf{Z}_{a,q}\langle \chi^\lambda \mid \lambda \vdash n \rangle$$

where χ_λ is the character of the irrep V_λ.

ann is roughly the cocenter map for the *Hecke algebra*.

For a certain choice of $(*)$, the maps

$$Br_n \xrightarrow{\text{ann}} \mathbf{Z}_{a,q}\langle \chi^\lambda \mid \lambda \vdash n \rangle \xrightarrow{\text{fill}} \mathrm{Sk}(\mathbf{R}^2)$$

can be related to geometric representation theory.

2 Varieties Let $G = \mathrm{PGL}_n$.

$$G = \coprod_{w \in S_n} BwB$$

where B is the image of upper-triangular matrices.

Given $\beta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_\ell} \in Br_n^+$ and $\pi \in S_n$:

The usual braid variety of (β, π) is

$$X(\beta,\pi) := \left\{ \vec{x}B \in O(\beta) \;\middle|\; \begin{array}{l} x_0 B = B, \\ x_\ell B = \pi B \end{array} \right\},$$

where $O(\beta) := \{x_0 B \xrightarrow{i_1} x_1 B \xrightarrow{i_2} \cdots \xrightarrow{i_\ell} x_\ell B\}$.

$X(\beta, \pi)$ is convenient for cell decompositions, cluster theory, explicit matrices, ...

We work instead with

$$\mathcal{V}(\beta) := \{\vec{x}B \in O(\beta) \mid x_0 B = x_\ell B\}.$$

Its defining condition is more visibly *annular*.

Writing $H_\pi := B \cap \pi B \pi^{-1}$ gives

$$[\mathcal{V}(\sigma_{\pi^{-1}}\alpha)/G] \simeq [X(\alpha, \pi)/H_\pi]$$

for any $\alpha \in Br_n^+$ and $\pi \in S_n$.

Let $\mathcal{P}(L) \in \mathbf{Z}_{a,q}$ be the HOMFLYPT of L, and

$$\mathcal{P}_k(L) = (-1)^k(\text{coefficient of } a^{\text{low}+2k} \text{ in } \mathcal{P}(L)).$$

(Shende–Treumann–Zaslow 2013) For $\beta = \sigma_{\pi^{-1}}\alpha$,

$$\begin{aligned} q^{(\ell(\beta)-n+1)/2}\,\mathcal{P}_{n-1}(\hat{\beta}) &= \chi_G(\mathcal{V}(\beta)) \\ &= \chi_{H_\pi}(X(\alpha, \pi)) \end{aligned}$$

where we write

$$\chi_H(X) = \frac{|X(\mathbf{F}_q)|}{|H(\mathbf{F}_q)|}.$$

The "highest" a-degree of $\mathcal{P}(\hat{\beta})$ is a point count.

What about the other a-degrees?

(Kálmán 2009) For the *full twist* Δ^2,

$$\mathcal{P}_0(\hat{\beta}) = \mathcal{P}_{n-1}(\widehat{\beta\Delta^2}).$$

Categorified by Gorsky–Hogancamp–Mellit–Nakagane.

(Bezrukavnikov–Tolmachov 2021) $\exists\, E_k \in \mathbf{Z}Br_n^+$ s.t.

$$\mathcal{P}_k(\hat{\beta}) = \mathcal{P}_{n-1}(\widehat{\beta E_k}).$$

(T.–Williams 2025) Explicitly,

$$E_k = \sum_{\substack{w \in S_n \\ s_i w < w \text{ iff } i \leq n-1-k}} \sigma_{w^{-1}} \sigma_w.$$

But tricky to categorify.

Recall: $\mathrm{Sk}_n(\mathbf{R}^2 - \mathbf{0}) \simeq \mathbf{Z}_{a,q}\langle \chi^\lambda \mid \lambda \vdash n \rangle$.

Each irrep V_λ occurs in the *Springer sheaf*

$$S_n \curvearrowright \mathcal{S}$$

over the unipotent variety $\mathcal{U} \subseteq G$.

$\mathcal{S}$ is the pushforward along the *Springer resolution*

$$\begin{array}{c} \widetilde{\mathcal{U}} := \{(u, yB) \mid uyB = yB\} \\ \downarrow \\ \mathcal{U} \end{array}$$

of the constant sheaf.

Recall $\mathcal{V}(\beta) = \{\vec{x}B \mid x_0 B = x_m B\} \subseteq O(\beta)$.

Enlarge this to a variety over $\mathcal{U}$:

$$\begin{aligned}\mathcal{U}(\beta) &:= \{(u, \vec{x}B) \mid ux_0 B = x_\ell B\} \\ &\subseteq \mathcal{U} \times O(\beta).\end{aligned}$$

The pullback along the Springer resolution is

$$\begin{aligned}\widetilde{\mathcal{U}}(\beta) &:= \left\{(u, \vec{x}B, yB) \;\middle|\; \begin{array}{l} ux_0 B = x_\ell B, \\ uyB = yB \end{array}\right\} \\ &\subseteq \mathcal{U} \times O(\beta) \times G/B.\end{aligned}$$

(Bezrukavnikov–Boixeda-Álvarez–McBreen–Yun 2022)

They're character varieties on $\mathbf{P}^1$ with two punctures, one wild and one tame.

(T. 2021)

1 ann categorified by

$$S_n \curvearrowright \mathrm{H}^*_{c,G}(\widetilde{\mathcal{U}}(\beta)).$$

2 fill categorified by

$$\mathsf{HHH}(\hat{\beta}) \propto \bigoplus_k \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,G}(\widetilde{\mathcal{U}}(\beta))[\chi^{(n-k,1^k)}]$$

where HHH is *triply-graded link homology* and $\mathsf{W}_\leq$ is the weight filtration.

Corollary (T.) For all k,

$$\mathcal{P}_k(\hat{\beta}) \propto \chi_G(\widetilde{\mathcal{U}}(\beta))[\chi^{(n-k,1^k)}].$$

Proof sketch

Webster–Williamson construct $\mathsf{HHH}(\hat{\beta})$ by applying $\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_G^*$ term-by-term to a certain complex

$$\mathcal{K}_\beta \in \mathsf{K}^b(\mathsf{Perv}_G^{\mathcal{G}}(G))$$

built from summands of the Grothendieck sheaf $\mathcal{G}$.

Their a is roughly $\mathrm{gr}_{i+*}^{\mathsf{W}} \mathrm{H}_G^i$.

I relate $\mathrm{H}_{c,G}^*(\widetilde{\mathcal{U}}(\beta))$ to $\mathcal{S} \overset{\mathrm{L}}{\otimes} \mathcal{K}_\beta|_{\mathcal{U}}$.

a becomes $[\chi^{(n-*,1^*)}]$ from comparing

$$\mathrm{H}_G^i(\mathcal{G}_\lambda) \quad \text{and} \quad \mathrm{H}_{c,G}^i(\mathcal{S} \otimes \mathcal{S}_\lambda)[\chi^{\mathrm{hooks}}].$$

3 Kawanaka's Identity

$\chi^{(n)}$ is trivial; $\chi^{(1^n)}$ is sign.

Using Springer theory, we can match:

$$\mathrm{H}_{c,G}^*(\widetilde{\mathcal{U}}(\beta))[\chi^{(n)}] \simeq \mathrm{H}_{c,G}^*(\mathcal{U}(\beta)),$$
$$\mathrm{H}_{c,G}^*(\widetilde{\mathcal{U}}(\beta\Delta^2))[\chi^{(1^n)}] \simeq \mathrm{H}_{c,G}^*(\mathcal{V}(\beta\Delta^2)).$$

(Gorsky–Hogancamp–Mellit–Nakagane 2019)

$$\widehat{\mathsf{HHH}_n(\beta\Delta^2)} \simeq \mathsf{HHH}_0(\hat{\beta}).$$

Therefore, $\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,G}^*(\mathcal{V}(\beta\Delta^2)) \simeq \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,G}^*(\mathcal{U}(\beta))$.

Example Take $\beta = \sigma_w$.

$$\mathcal{U}(\sigma_w) \simeq (G \times (\mathcal{U} \cap BwB))/B,$$
$$\mathcal{V}(\sigma_w \Delta^2) \simeq (G \times (UU_- \cap BwB))/B,$$

where $U = B \cap \mathcal{U}$ and $U_- = B_- \cap \mathcal{U}$.

For $G = \mathrm{PGL}_3$ and $w = w_\circ$, not even homeomorphic.

Conjecture (T. 2022) The map

$$UU_- \cap BwB \to \mathcal{U} \cap BwB$$
$$uv \mapsto uvu^{-1}$$

is a *homotopy equivalence* over $\mathbf{C}$.

Isomorphism on $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,G}$ by Gysin.

Why do I care?

(Fine–Herstein 1958, Steinberg 1965)

$$|\mathcal{U}(\mathbf{F}_q)| = q^{n(n-1)} = |UU_-(\mathbf{F}_q)|.$$

(Kawanaka 1975)

$$|(\mathcal{U} \cap BwB)(\mathbf{F}_q)| = |(UU_- \cap BwB)(\mathbf{F}_q)|.$$

Kawanaka's proof used representation theory of $G(\mathbf{F}_q)$. Unknown how to categorify.

Everything here generalizes to (split) reductive G with Weyl group W.

4 Deodhar's Cells

Do $\mathcal{V}(\beta)$, $\mathcal{U}(\beta)$, $\widetilde{\mathcal{U}}(\beta)$ have toric charts or cells?

Pass to stack quotients:

$$\begin{aligned} \text{just as} \quad & [\mathcal{V}(\sigma_{\pi^{-1}}\alpha)/G] \simeq [X(\alpha,\pi)/H_\pi], \\ \text{so} \quad & [\widetilde{\mathcal{U}}(\sigma_{\pi^{-1}}\alpha)/G] \simeq [\widetilde{X}(\beta)/B], \end{aligned}$$

where we set

$$\widetilde{X}(\beta) := \{(u, \vec{x}B) \in U \times O(\beta) \mid ux_0B = x_\ell B\}.$$

Note: $u \in U$, not just $u \in \mathcal{U}$.

(T.–Williams 2025) $\widetilde{X}(\beta) = \coprod_{w\in W} \widetilde{X}_w(\beta)$, where

$$\widetilde{X}_w(\beta) := \{(u, \vec{x}B) \in \widetilde{X}(\beta) \mid Bx_\ell B = BwB\}$$

is an $\mathbf{A}^{\ell(w_\circ)-\ell(w)}$-bundle over

$$X(\sigma_w\beta, w).$$

In this sense, $\widetilde{X}(\beta)$ has a Deodhar decomposition.

Is it a cluster variety?

Corollary (T.–Williams)

$$|\widetilde{\mathcal{U}}(\beta)(\mathbf{F}_q)| \propto \sum_{w\in W} q^{-\ell(w)} |\mathcal{V}(\sigma_{w^{-1}}\sigma_w\beta)(\mathbf{F}_q)|.$$

The *Harish-Chandra transform* is $\mathsf{hc} = act_! pr^*$ in

$$G \xleftarrow{pr} G \times G/B \xrightarrow{act} G/B \times G/B.$$

On $\mathbf{F}_q$-points, the RHS gives the Hecke algebra.

Class functions on $G(\mathbf{F}_q)$ become *central* Hecke elements.

The preceding theorem can be recast as

$$[\mathsf{hc}(\mathcal{S})] \propto \sum_{w \in W} \sigma_{w^{-1}} \sigma_w.$$

(T.–Williams 2025) Two parabolic generalizations.

Mutually dual via Borho–MacPherson.

Dichotomy between U_J and $\mathcal{U}_J$ within $P_J \subseteq G$.

Two partial Springer resolutions $\mathcal{S}_J^{\pm}$.

$$[\mathsf{hc}(\mathcal{S}_J^{\pm})] \propto \sum_{w \in W^{J,\pm}} \sigma_{w^{-1}} \sigma_w.$$

Only the $-$ case gives a Deodhar decomposition!

The $+$ case is harder; it requires Kawanaka.

Thank you for listening.