Triply-Graded Link Homology and the Hilb-vs-Quot Conjecture

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Plan of talk:

1. Links of Plane Curves
2. The Quot Schemes $\mathcal{Q}_f^\ell$
3. Combinatorics
4. The Variety $\overline{\mathcal{P}}_f/\Lambda_f$

References

- M. Q. Trinh. From the Hecke Category to the Unipotent Locus (2021). 88 pp. arXiv:2106.07444
- O. Kivinen &. M. Q. Trinh. The Hilb-vs-Quot Conjecture (2023). 51 pp. arXiv:2310.19633

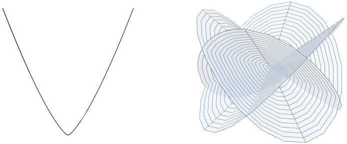
1. Links of Plane Curves

$$\{f(x,y)=0\} \quad \text{for nonzero, squarefree } f \in \mathbf{C}[x,y] \text{ with } f(0,0)=0$$

Ex $\{y=0\}$ is smooth.

Ex $\{xy=0\}$ has a node at $(0,0)$.

Ex $\{y^3=x^4\}$ has a cusp at $(0,0)$.



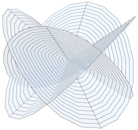
LHS in $\mathbf{R}^2$. RHS in $\mathbf{C}^2$, but projected into $\mathbf{R}^3$.

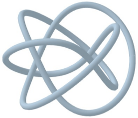
For $\varepsilon > 0$, the preimage of the circle $|x| = \varepsilon$ in $\{y^3 = x^4\}$ is the *torus knot* $T_{3,4}$.

In general, a plane curve germ $f = 0$ gives rise to

$$\text{a topological link } L_f \subseteq S^3.$$

If $f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y + a_n(x)$, then L_f is the *closure* of a braid on n strands.





What aspects of the geometry of $f = 0$ only depend on L_f?

Let $R_f = \mathbf{C}[\![x, y]\!]/(f)$, the completed local ring.

Consider the *punctual Hilbert schemes*

$$\mathcal{H}_f^\ell = \{\text{ideals } I \subseteq R_f \mid \dim_{\mathbf{C}}(R_f/I) = \ell\}.$$

Ex If f is smooth, then $\mathcal{H}_f^\ell$ is a point for all ℓ.

Ex If $f(x, y) = xy$, then $\mathcal{H}_f^0, \mathcal{H}_f^1$ are points, but

$$\mathcal{H}_f^2 = \{\langle x^2, y\rangle, \langle x, y^2\rangle\} \cup \{\langle x + \lambda y\rangle\} \simeq \mathbf{CP}^1.$$

In general, $\mathcal{H}_f^\ell$ is a chain of transverse $\mathbf{CP}^1$'s.

We'll also use *nested Hilbert schemes*

$$\mathcal{H}_f^{\ell,m} = \{(I, J) \in \mathcal{H}_f^\ell \times \mathcal{H}_f^{\ell+m} \mid \langle x, y\rangle I \subseteq J \subseteq I\}.$$

Picture as a stratified fiber bundle over $\mathcal{H}_f^\ell$.

Consider the *virtual weight polynomial*

$$\chi(-,t) : K_0(\mathsf{Var}_{\mathbf{C}}) \to \mathbf{Z}[t].$$

Explicitly, $\chi(X,t) = \sum_{i,j} (-1)^i t^j \dim \mathrm{gr}_j^{\mathsf{W}}\, \mathrm{H}_c^i(X)$.

In practice, use these facts:

$$\chi(\mathbf{CP}^1, t) = 1 + t^2,$$
$$\chi(X,t) = \chi(Z,t) + \chi(X - Z, t),$$
$$\chi(X \times Y, t) = \chi(X,t)\chi(Y,t).$$

We'll study $\sum_\ell q^\ell \chi(\mathcal{H}_f^\ell, t)$. Also a nested version.

Ex For $f(x,y) = xy$, get

$$\sum_\ell q^\ell \chi(\mathcal{H}_f^\ell, t) = \frac{1}{(1-q)^2}(1 - q + q^2 t^2).$$

In general, the denominator of $\sum_\ell q^\ell \chi(\mathcal{H}_f^\ell, t)$ looks like $(1-q)^b$.

In fact, b is just the number of components of L_f.

Something stronger. There's a quantum link invariant called *Khovanov–Rozansky (KhR) homology*:

$$\mathsf{KhR} : \{\text{links up to isotopy}\} \to \mathsf{Vect}_{3\text{-gr}}.$$

Its graded dimension is a series in a, q, t. We'll use a normalization $\bar{X}$ such that $\bar{X}_{\text{unknot}} = \frac{1+a}{1-q}$.

Conj (Oblomkov–Rasmussen–Shende)

$$\bar{X}_{L_f}(a, q, qt^2) = \sum_{\ell,m} q^\ell a^m t^{m(m-1)} \chi(\mathcal{H}_f^{\ell,m}, t).$$

Thm (Maulik) The conjecture holds at $t=-1$.

Proof used flop identities in the resolved conifold.

Note that at $t=-1$, KhR homology specializes to the *HOMFLYPT polynomial* $\bar{P}_L$ given by

$$a\bar{P}_{\nwarrow\!\!\!\nearrow} - a^{-1}\bar{P}_{\nwarrow\!\!\!\nearrow} = (q^{1/2}-q^{-1/2})\bar{P}_{)(},$$

$$\bar{P}_{\text{unknot}} = \frac{a-a^{-1}}{q^{1/2}-q^{-1/2}}.$$

One computes KhR through a *categorification* of these skein relations.

Not known how to categorify Maulik's flop identities.

For experts: If L is the closure of an n-strand braid of writhe e, then $\bar{P}_L(a,q) = (aq^{-1})^{e-n}\bar{X}_L(-a,q,q)$.

Prop (ORS) The conjecture holds for $y^2 = x^d$ with d odd.

Here, the link is the torus knot $T_{2,d}$. Proof used an asymptotic formula for the $\mathcal{H}_f$ side.

Also, for $3 \nmid d$, a conjectural formula for $\bar{X}_{T_{3,d}}$ implies the ORS conjecture for $y^3 = x^d$.

In general, L_f is merely an *iterated torus link*.

Ex If $f(x,y) = xy$, then L_f is a *Hopf link* .

$$\bar{X}_{L_f}(a,q,qt^2) = \frac{1}{1-q} + \frac{(q+a)(qt^2)}{(1-q)^2}$$

At $a=0$, we do recover

$$\frac{1-q+q^2t^2}{(1-q)^2} = \sum_{\ell} q^{\ell}\chi(\mathcal{H}_f^{\ell}, t).$$

2. The Quot Schemes $\mathcal{Q}_f^\ell$

We'll factor the ORS conjecture into two statements.

Recall $b = |\pi_0(L_f)|$ = number of branches of f. Let

$$\tilde{R}_f := \mathbf{C}[\![t_1]\!] \times \cdots \times \mathbf{C}[\![t_b]\!]$$

Have a *normalization* map $R_f \hookrightarrow \tilde{R}_f$.

Ex If $f(x, y) = xy$, then

$$R_f \simeq \mathbf{C}[\![t_1, t_2]\!]/(t_1 t_2), \qquad \tilde{R}_f \simeq \mathbf{C}[\![t_1]\!] \times \mathbf{C}[\![t_2]\!].$$

Ex If $f(x, y) = y^3 - x^4$, then

$$R_f \simeq \mathbf{C}[\![t^3, t^4]\!], \qquad \tilde{R}_f \simeq \mathbf{C}[\![t]\!].$$

Consider the *Quot schemes* of $\tilde{R}_f$ as an R_f-module:

$$\mathcal{Q}_f^\ell = \{\text{submodules } M \subseteq \tilde{R}_f \mid \dim_{\mathbf{C}} \tilde{R}_f/M = \ell\}.$$

There's a nested version $\mathcal{Q}_f^{\ell,m}$ analogous to $\mathcal{H}_f^{\ell,m}$.

Kivinen and I propose: It's easier to relate the $\mathcal{Q}_f^{\ell,m}$ to KhR homology. For $\mathcal{S} \in \{\mathcal{H}, \mathcal{Q}\}$, let

$$Z_f^{\mathcal{S}}(a, q, t) = \sum_{\ell,m} q^\ell a^m t^{m(m-1)}\, \chi(\mathcal{S}_f^{\ell,m}, t).$$

Conj (Kivinen–T)

$$\bar{X}_{L_f}(a, q, qt^2) \overset{(1)}{=} Z_f^{\mathcal{Q}}(a, q, q^{1/2}t) \overset{(2)}{=} Z_f^{\mathcal{H}}(a, q, t).$$

(1) = "Quot ORS". (2) = "Hilb-vs-Quot".

Thm (KT) Quot ORS holds for any $y^n = x^d$ with:

(1) n, d coprime.
(2) n dividing d.

Thm (KT) Hilb-vs-Quot holds asymptotically for $y^n = x^d$ with d coprime to n, in the $d \to \infty$ limit.

Cor Hilb-vs-Quot holds for $y^3 = x^d$ with $3 \nmid d$.

Cor The original ORS conjecture holds for $y^3 = x^d$ with $3 \nmid d$.

Thus the conjectural closed formula for $\bar{X}_{T_{3,d}}$ is true.

Rest of this talk: Explain why $y^n = x^d$ is easier. Explain implications for *nonabelian Hodge theory*.

3. Combinatorics Key point:

For $y^n = x^d$ with n, d coprime, the schemes $\mathcal{H}_f^\ell, \mathcal{Q}_f^\ell$ are *paved by affine spaces*.

Each paving stratum is centered at an ideal/module generated by monomials in x, y.

If M is such a module, then its stratum is

$$\mathbf{C}_M = \{N \mid \lim_{t \to 0} t \cdot N = M\},$$

where $c \in \mathbf{C}^\times$ acts by $c \cdot x = c^n x$ and $c \cdot y = c^d y$.

That is: the locus contracting to M under the flow induced by the $\mathbf{C}^\times$-action on $y^n = x^d$.

For experts: Not quite Białynicki-Birula because $\mathcal{H}_f^\ell, \mathcal{Q}_f^\ell$ could be singular.

Ex If $f(x,y) = y^3 - x^4$, then

$$R_f \simeq \mathbf{C}[\![t^3, t^4]\!] \quad \text{via } x = t^3 \text{ and } y = t^4.$$

The $\mathbf{C}^\times$-action is $c \cdot t = ct$.

$\mathcal{H}_f^0$	$\mathcal{H}_f^1$	$\mathcal{H}_f^2$	$\mathcal{H}_f^3$	$\mathcal{H}_f^4$	$\mathcal{H}_f^5$	$\mathcal{H}_f^6$
pt_0			$\mathbf{C}_3^2$	$\mathbf{C}_4^2$		$\mathbf{C}^3$
		$\mathbf{C}_{3,8}$	$\mathbf{C}_{4,9}$		$\mathbf{C}_{6,11}^2$	$\mathbf{C}^2$
	$pt_{3,4}$			$\mathbf{C}_{6,7}^2$	$\mathbf{C}_{7,8}^2$	$\mathbf{C}^2$
		$pt_{4,6}$		$\mathbf{C}_{6,8}$	$\mathbf{C}_{7,9}$	$\mathbf{C}$
			$pt_{6,7,8}$	$pt_{7,8,9}$	$pt_{8,9,10}$	pt

Above, $\mathbf{C}_{6,7}^2$ means $\mathbf{C}_{\langle t^6, t^7 \rangle} \simeq \mathbf{C}^2$ in $\mathcal{H}_f^4$.

Rows indicate R_f-module isomorphism types.

Colors show which strata can be reassembled into $\mathcal{Q}_f^0, \mathcal{Q}_f^1, \mathcal{Q}_f^2, \mathcal{Q}_f^3, \ldots$

Upshot: Can express $Z_f^{\mathcal{H}}, Z_f^{\mathcal{Q}}$ purely combinatorially.

Let $\Gamma = n\mathbf{Z}_{\geq 0} + d\mathbf{Z}_{\geq 0}$. For any $S \subseteq \mathbf{Z}_{\geq 0}$, let

$$I^\ell(S) = \{\Delta \subseteq S \mid \Gamma + \Delta \subseteq \Delta \text{ and } |S \setminus \Delta| = \ell\}.$$

The map $\Delta \mapsto \langle t^n \mid n \in \Delta \rangle$ yields bijections

$$I^\ell(\Gamma) \xrightarrow{\sim} \{\mathbf{C}^\times\text{-fixed points of } \mathcal{H}_f^\ell\},$$
$$I^\ell(\mathbf{Z}_{\geq 0}) \xrightarrow{\sim} \{\mathbf{C}^\times\text{-fixed points of } \mathcal{Q}_f^\ell\}.$$

Leads to formulas that look like

$$Z_f^{\mathcal{H}} = \sum_\ell \sum_{\Delta \in I^\ell(\Gamma)} q^\ell t^{2\,\mathsf{codinv}_{\mathcal{H}}(\Delta)} \Pi_n(\Delta, a, t),$$
$$Z_f^{\mathcal{Q}} = \sum_\ell \sum_{\Delta \in I^\ell(\mathbf{Z}_{\geq 0})} q^\ell t^{2\,\mathsf{codinv}_{\mathcal{Q}}(\Delta)} \Pi_n(\Delta, a, t)$$

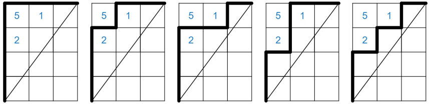
for certain functions codinv and $\Pi_n(-, a, t)$.

Sketch of Quot ORS for $y^n = x^d$ with n, d coprime

Can rewrite $Z_f^{\mathcal{Q}}$ (but not $Z_f^{\mathcal{H}}$!) in terms of a sum over

$$\{\Delta \subseteq \mathbf{Z}_{\geq 0} \mid \Gamma + \Delta \subseteq \Delta \text{ and } 0 \in \Delta\}$$

Gorsky–Mazin gave a bijection between these Δ and *(n, d) Dyck paths*:



Rewrite $\mathsf{codinv}_{\mathcal{Q}}, \Pi_n(-, a, t)$ in terms of Dyck-path invariants well-studied in Macdonald theory.

Compare to the *Gorsky–Neguţ formula* for $\mathsf{KhR}(T_{n,d})$ using Dyck paths, proved by Mellit.

Sketch of Hilb-vs-Quot for $y^3 = x^d$ with $3 \nmid d$

Using the Δ's, ORS computed $\lim_{d \to \infty} Z_f^{\mathcal{H}}$.

The key is to *sort* pairs of nested ideals $I \supseteq J$ in terms of a third, auxiliary ideal $I' \supseteq I$.

Computing a similar formula for $\lim_{d \to \infty} Z_f^{\mathcal{Q}}$ is even easier: no auxiliary module.

$Z_f^{\mathcal{H}}, Z_f^{\mathcal{Q}}$ match the limit formulas up to $O(q^d)$.

Let $\delta = \frac{1}{2}(n-1)(d-1)$. As we'll explain:

(1) $q^{-\delta}(1-q)Z_f^{\mathcal{H}}$ has a $q^{-1} \leftrightarrow qt^2$ symmetry.

(2) $q^{-\delta}(1-q)Z_f^{\mathcal{Q}}$ has a $q^{-1} \leftrightarrow t^2$ symmetry.

So we can compute $Z_f^{\mathcal{H}}, Z_f^{\mathcal{Q}}$ as long as $\delta < d$.

Rem Why should there be a formula for $\mathsf{KhR}(T_{n,d})$ in terms of (n,d) Dyck paths?

In general, KhR of iterated torus links should emerge from *shuffle operators* in an elliptic Hall algebra acting on q,t-Fock space. The *Carlsson–Mellit theorem* gives formulas for this action via sums over Dyck paths.

Rem Why does $q^{-\delta}(1-q)Z_f^{\mathcal{Q}}$ have symmetry?

The $q^{1/2} \leftrightarrow -q^{-1/2}$ symmetry of HOMFLYPT lifts to KhR (for knots), as proved by Oblomkov–Rozansky. Now invoke Quot ORS.

Rem Why does $q^{-\delta}(1-q)Z_f^{\mathcal{H}}$ have symmetry?

Compare $Z_f^{\mathcal{H}}$ to an analogue for a projective rational curve. There the symmetry comes from Serre duality.

4. The Variety $\overline{\mathcal{P}}_f/\Lambda_f$

For general f, not of the form $y^n = x^d$, the series

$$q^{-1/2-\delta}(q^{-1/2} - q^{1/2})^b Z_f^{\mathcal{H}}$$

still satisfies palindromic symmetry.

It can be viewed not just as Serre duality, but as a *perverse hard Lefschetz symmetry* for a new variety.

Note that $\mathrm{Frac}(R_f) = \mathbf{C}(\!(t_1)\!) \times \cdots \times \mathbf{C}(\!(t_b)\!)$. Let

$$\overline{\mathcal{P}}_f = \left\{ \begin{array}{l} \text{finite-type} \\ M \subseteq \mathrm{Frac}(R_f) \end{array} \middle| \, \mathrm{Frac}(R_f)M = \mathrm{Frac}(R_f) \right\}.$$

Let $\Lambda_f = \{t_1^{e_1} \cdots t_b^{e_b} \mid \vec{e} \in \mathbf{Z}\} \curvearrowright \overline{\mathcal{P}}_f$.

It turns out that $\overline{\mathcal{P}}_f/\Lambda_f$ is a projective variety.

$\overline{\mathcal{P}}_f$ is the analogue, for R_f, of the *compactified Picard* of an integral, locally planar projective curve.

It is a space of "possibly degenerating line bundles" on the germ $\mathrm{Spec}(R_f)$.

Laumon and others observed: If $f(x, y) = 0$ is a branched n-fold cover of the x-axis, so that

$$R_f \simeq \mathbf{C}[\![x]\!]^{\oplus n} \quad \text{as } \mathbf{C}[\![x]\!]\text{-modules},$$

then any point of $\overline{\mathcal{P}}_f$ also forms a $\mathbf{C}[\![x]\!]$-submodule of $\mathbf{C}(\!(x)\!)^{\oplus n}$ with a linear operator y.

Geometrically: a rank-n vector bundle on $\mathrm{Spec}(\mathbf{C}[\![x]\!])$ with an operator y, framed on the punctured disk.

A vector bundle with a (twisted) operator is a *Higgs bundle*. Generalizes to other reductive G.

Hitchin observed that moduli of Higgs bundles are fibered by (possibly degenerating) Lagrangian tori.

$$\overline{\mathcal{P}}_f \quad \text{“}\hookrightarrow\text{”} \quad \mathcal{M}_{\mathrm{H}} \xrightarrow{h} \mathcal{A}_{\mathrm{H}}.$$

Ngô used the decomposition of the sheaf complex

$$\mathrm{R}h_*\mathbf{C}$$

into perverse sheaves to prove the Fundamental Lemma for orbital integrals on $\mathrm{Lie}(G)$.

Upshot for $\overline{\mathcal{P}}$: An increasing *perverse filtration*

$$\mathsf{P}_{\leq *} \quad \text{on} \quad \mathrm{H}^*(\overline{\mathcal{P}}_f/\Lambda_f).$$

For any X and filtration F on $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_c(X)$, let

$$\chi^{\mathsf{F}}(X, q, t) = \textstyle\sum_{i,j,k} (-1)^i q^j t^k \dim \mathrm{gr}^{\mathsf{F}}_j \mathrm{gr}^{\mathsf{W}}_k \mathrm{H}^i_c(X).$$

Thm (Maulik–Yun, Migliorini–Shende)

$$\frac{\chi^{\mathsf{P}}(\overline{\mathcal{P}}_f/\Lambda_f, q, t)}{(1-q)^b} = \sum_\ell q^\ell \chi(\mathcal{H}^\ell_f, t).$$

Thm (Kivinen–T)

$$\frac{\chi^{\mathsf{Q}}(\overline{\mathcal{P}}_f/\Lambda_f, q, t)}{(1-q)^b} = \sum_\ell q^\ell \chi(\mathcal{Q}^\ell_f, t),$$

for the filtration $\mathsf{Q}^{\geq c}$ on $\mathrm{H}^*(\overline{\mathcal{P}}_f/\Lambda_f)$ induced by

$$\overline{\mathcal{P}}_{f,\leq c} := \{M \in \overline{\mathcal{P}}_f \mid \dim_{\mathbf{C}}(\tilde{R}_f M)/M \leq c\}.$$

Hilb-vs-Quot would imply $\chi^{\mathsf{P}}(q,t) = \chi^{\mathsf{Q}}(q, q^{\frac{1}{2}} t)$.

Nonabelian Hodge theory concerns non-algebraic homeomorphisms between the spaces $\mathcal{M}_{\mathrm{H}}$ and *Betti moduli spaces* of local systems.

In 2013, Shende speculated that

$$\mathsf{P}_{\leq *} \mathrm{H}^*(\overline{\mathcal{P}}_f/\Lambda_f) \simeq \mathsf{W}_{\leq */2} \mathrm{H}^*(\mathcal{M}_{\mathrm{B}})$$

for some *wild* Betti moduli space $\mathcal{M}_{\mathrm{B}}$ of *Stokes local systems*.

In 2021, I constructed a candidate for $\mathcal{M}_{\mathrm{B}}$ and an explicit embedding $\mathsf{KhR}(L_f) \hookrightarrow \mathsf{W}_{\leq */2} \mathrm{H}^*(\mathcal{M}_{\mathrm{B}})$.

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Thank you for listening.