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# Webs, Hecke Characters, and Springer's Correspondence

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## 1 Diagrams

Let $\mathbf{A} = \mathbf{Z}[a^{\pm 1}, q^{\pm 1/2}, \frac{1}{q^{1/2}-q^{-1/2}}]$.

The HOMFLYPT skein module of a surface $S$:

$$\mathrm{Sk}(S) = \mathbf{A}\langle \mathcal{L}(S) \rangle / Rel(S).$$

$\mathcal{L}(S)$ is generated by oriented link diagrams.

$Rel(S)$ is generated by

$$\bigcirc = \frac{a - a^{-1}}{q^{1/2} - q^{-1/2}},$$

$$a \; \diagup\!\!\diagdown \; - a^{-1} \; \diagdown\!\!\diagup \; = (q^{1/2} - q^{-1/2}) \; )\!(.$$

(≈ HOMFLYPT 1985) $\mathrm{Sk}(\mathbf{R}^2) \simeq \mathbf{A}$.

Sends each oriented link to an $a, q^{1/2}$-invariant.

How about $S = \mathbf{R}^2 - \mathbf{0}$?

Cannot simplify loops that go around $\mathbf{0}$.

Pairwise distinct diagrams $p_k$ for $k \in \mathbf{Z}$.

$p_5 =$ 

Commutative ring structure via stacking diagrams.

(Turaev 1988) As an algebra,

$$\mathrm{Sk}(\mathbf{R}^2 - \mathbf{0}) \simeq \mathbf{A}[p_{\pm 1}, p_{\pm 2}, p_{\pm 3}, \ldots].$$

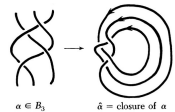
That is, an infinite-type polynomial algebra over $\mathbf{A}$.

Let $Br_n$ be the braid group on $n$ strands.

The diagrammatic *Hecke algebra* is

$$H_n = \frac{\mathbf{Z}[q^{\pm 1/2}][Br_n]}{\langle \times - \times - (q^{1/2} - q^{-1/2}) \rangle}.$$

The *annular closure* operation



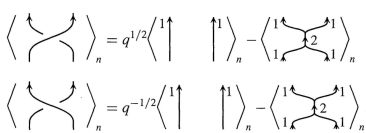

defines a trace map

$$\mathsf{ann}\colon H_n \to \mathrm{Sk}(\mathbf{R}^2 - \mathbf{0}).$$

It factors through $\mathbf{A}[p_1, p_2, p_3, \ldots]$.

(Murakami–Ohtsuki–Yamada 1998)

New generators for $H_n$ from colored *webs*:



The $i$th ⌶ generator: $C_i = q^{1/2} - \sigma_i = q^{-1/2} - \sigma_i^{-1}$.

In general webs, vertices are *trivalent* and obey *conservation of color*.

How to picture their annular closures?

$\mathbf{Z}[q^{\pm 1/2}]$-linear combinations of *colored loops*.

Let $e_n$ be a single loop with winding number 1 and color $n$.

$\mathbf{A}[p_1, p_2, p_3, \ldots]$ is, secretly, the *ring of symmetric functions* over $\mathbf{A}$, with bases

$$\begin{aligned} \textit{powersum} \quad & \{p_\mu = p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} \ldots\}_\mu \\ \textit{elementary} \quad & \{e_\mu = e_1^{\mu_1} e_2^{\mu_2} e_3^{\mu_3} \ldots\}_\mu \end{aligned}$$

indexed by integer partitions $\mu$.

Conj (Folklore) Any annular web

$$\mathsf{ann}(C_{i_1} C_{i_2} \cdots C_{i_\ell})$$

has a positive ( $= \mathbf{Z}_{\geq 0}[q^{\pm 1/2}]$-linear) expansion in $\{e_\mu\}_\mu$.

At the same time, the $c_i$ are secretly elements of a *canonical basis* for $H_n$ indexed by $z \in S_n$.

(Kazhdan–Lusztig 1979) Bases for $H_n$ defined by

$$C_z = \sum_{y \le z} (-1)^{\ell_y} q^{(\ell_z - \ell_y)/2} P_{y,z}(q^{-1}) \sigma_y,$$
$$C'_z = \sum_{y \le z} q^{(\ell_y - \ell_z)/2} P_{y,z}(q) \sigma_y,$$

where the polynomials are uniquely determined by $P_{z,z}(q) = 1$ and $\deg P_{y,z} \le \frac{1}{2}(\ell_z - \ell_y - 1)$.

These bases are fixed by the ring anti-involution sending $q^{1/2} \mapsto q^{-1/2}$ and $\sigma_i \mapsto \sigma_i^{-1}$.

By contrast, the involution sending $q^{1/2} \mapsto q^{-1/2}$ and $\sigma_i \mapsto -\sigma_i$ swaps $C_z$ with $C'_z$.

Under $H_n \xrightarrow{\mathsf{ann}} \mathbf{A}[p_1, p_2, p_3, \ldots]$, it swaps

elementary $\{e_\mu = e_1^{\mu_1} e_2^{\mu_2} e_3^{\mu_3} \ldots\}_\mu$

with

*complete homogeneous* $\{h_\mu = h_1^{\mu_1} h_2^{\mu_2} h_3^{\mu_3} \ldots\}_\mu$.

Conj ($\approx$ Haiman 1993) For any $z \in S_n$,

$$\mathsf{ann}(C'_z) \quad (\textit{resp.}\ \mathsf{ann}(C_z))$$

has a positive expansion in the $h_\mu$ (*resp.* $e_\mu$) for $\mu \vdash n$.

Subsumes the folklore conjecture.

In fact, Haiman's original formulation used characters, not diagrams.

The ring of symmetric functions is also

$$\bigoplus_{n\geq 1} \mathbf{A}\langle \mathrm{Irr}(S_n)\rangle$$

under the product $\chi \cdot \psi := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(\chi \boxtimes \psi)$.

$$p_\mu \leftrightarrow \text{rescaled indicator on cycle type } \mu'$$
$$e_\mu \leftrightarrow \mathrm{Ind}_{S_\mu}^{S_n}(\mathsf{sgn})$$
$$h_\mu \leftrightarrow \mathrm{Ind}_{S_\mu}^{S_n}(1)$$
*Schur* $$s_\mu \leftrightarrow \text{irreducible } \chi^\mu$$

What is annular closure in terms of characters?

For all $\beta \in H_n$,

$$\mathsf{ann}(\beta) = \sum_{\lambda \vdash n} \chi_q^\lambda(\beta) s_\lambda,$$

where $\chi_q^\lambda \in \mathrm{Irr}(H_n)$ deforms $\chi^\lambda \in \mathrm{Irr}(S_n)$.

Conj (Haiman 1993) For any $z \in S_n$, if we expand

$$\mathsf{ann}(C_z') = \sum_{\mu \vdash n} \alpha_\mu^z h_\mu \quad \text{for some } \alpha_\mu^z \in \mathbf{Z}[q^{\pm 1/2}],$$

then the nonzero coefficients of $\alpha_\mu^z$ are positive and unimodal for all $\mu$.

Haiman checked this up through $n = 7$.

Why is this is hard?

The $h_\mu$ expand positively in the $s_\lambda$, but not vice versa.

**Example** For $z = s_2 s_1 s_3 s_2 \in S_4$,

$$\begin{aligned} \mathsf{ann}(C'_z) = &\ ([\mathbf{5}] + 3\cdot[\mathbf{3}] + 2\cdot[\mathbf{1}])s_4 & ([\mathbf{5}] + [\mathbf{3}])h_4 \\ &+ (2\cdot[\mathbf{3}] + 2\cdot[\mathbf{1}])s_{31} & = +[\mathbf{3}]h_{31} \\ &+ ([\mathbf{3}] + [\mathbf{1}])s_{22} & +[\mathbf{3}]h_{22} \end{aligned}$$

where $[\boldsymbol{m}] := \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$.

(Hikita, Griffin–Mellit–Romero–Weigl–Wen 2025)

True for 312-*avoiding* permutations $z$ at $q^{1/2} \to 1$.

(The Stanley–Stembridge conjecture.)

How to generalize Haiman's conjecture beyond $S_n$?

It is about $\alpha_\mu^z \in \mathbf{Z}[q^{\pm 1/2}]$ defined by

$$\sum_{\chi \in \mathrm{Irr}(S_n)} \chi_q(C'_z)\chi = \sum_{\mu \vdash n} \alpha_\mu^z \operatorname{Ind}_{S_\mu}^{S_n}(1).$$

Can we generalize from $S_n$ to, say, Weyl groups?

**Issue 1** Beyond $S_n$, even the positivity of $\chi_q(C'_z)$ for irreducible $\chi$ can fail.

**Issue 2** Beyond $S_n$, not enough conjugacy classes of parabolic subgroups to replace the $S_\mu$.

We attempt to solve these issues through geometry.

2 Varieties Let $G$ be a complex reductive group.

Let $W$ be its Weyl group.

Let $B \subseteq G$ be a fixed Borel.

*Springer theory* tells us to build representations of $W$ from the cohomology of the *Springer fibers*

$$\mathcal{B}_g = \{xB \in G/B \mid x^{-1}gx \in B\}.$$

Example Let $G^{rs}$ be the regular semisimple locus.

If $G = \mathrm{GL}_n$, then $G^{rs} = \{\text{multiplicity-1 eigenvalues}\}$.

For any $g \in G^{rs}(\mathbf{C})$, the Springer fiber is a $W$-torsor.

(Springer 1978) For any $g \in G(\mathbf{C})$, still have

$$W \curvearrowright \mathrm{H}^*(\mathcal{B}_g).$$

In fact, an injective map

$$\mathrm{Irr}(W) \hookrightarrow \left\{ (u, \kappa) \;\middle|\; \begin{array}{l} u \in G(\mathbf{C}) \text{ unipotent}, \\ \kappa \in \mathrm{Irr}(\pi_0(G_u)) \end{array} \right\} \Big/ \sim$$

where $\sim$ is conjugacy:

$$\chi \mapsto [u, \kappa] \quad \Longleftrightarrow \quad \chi \text{ occurs in } \mathrm{H}^{\mathrm{top}}(\mathcal{B}_u)_\kappa.$$

Example For $G = \mathrm{GL}_n$ and $u$ unipotent of type $\mu$:

$$\chi^\mu \text{ occurs in } \mathrm{H}^*(\mathcal{B}_u) \simeq \mathrm{Ind}_{S_\mu}^{S_n}(1).$$

A generalization of Springer fibers:

For any $z \in W$, the *closed Lusztig variety*

$$\mathcal{Y}_{z,g} = \{xB \in G/B \mid x^{-1}gx \in \overline{BzB}\}.$$

For $g \in G^{rs}(\mathbf{C})$, monodromy of $\pi_1(G^{rs})$ descends to

$$W \curvearrowright \mathrm{IH}^*(\mathcal{Y}_{z,g}).$$

Thm (T. 2026, using Lusztig + Abreu–Nigro)

$$\sum_i q^{(i-\ell_z)/2} \mathrm{IH}^i(\mathcal{Y}_{z,g}) = \sum_{\chi,\psi \in \mathrm{Irr}(W)} \{\chi,\psi\}\psi_q(C'_z)\chi$$

where $\{-,-\}$ is Lusztig's exotic Fourier transform.

For $G = \mathrm{GL}_n$, r.h. sum is Haiman's $\sum_\chi \chi_q(C'_z)\chi$.

Issue 1 Beyond $S_n$, even the positivity of $\chi_q(C'_z)$ for irreducible $\chi$ can fail.

Issue 2 Beyond $S_n$, not enough conjugacy classes of parabolic subgroups to replace the $S_\mu$.

Solutions (T., Brosnan–J. Hong–D. Lee–E. Lee 2026)

1 Replace $\sum_\chi \chi_q(C'_z)\chi$ with

$$\sum_{\chi,\psi \in \mathrm{Irr}(W)} \{\chi,\psi\}\psi_q(C'_z)\chi.$$

2 Replace the $\mathrm{Ind}_{S_\mu}^{S_n}(1)$ with

$$spr_\psi = \mathrm{H}^*(\mathcal{B}_u)_\kappa \quad \text{for } \psi \in \mathrm{Irr}(W),$$

where $(u,\kappa)$ is the Springer image of $\psi$.

## 3 Calculations

So beyond type $A$:

Study the nonzero coefficients of the $\alpha^z_{\psi,G} \in \mathbf{Z}[q^{\pm 1/2}]$ defined by

$$\sum_{\chi,\psi \in \mathrm{Irr}(W)} \{\chi,\psi\}\psi_q(C_z')\chi = \sum_{\psi \in \mathrm{Irr}(W)} \alpha^z_{\psi,G} spr_\psi.$$

Can use the GAP3 package CHEVIE to compute lots of examples.

! Although positivity often holds, it can still fail, starting in type $B_2$ where there is a single failure.

! Even up to absolute value, unimodality can fail, starting in type $C_4$.

Example The $\alpha^z_{\psi,G}$ in type $G_2$.

| | $e$ | 2 | 1 | 21, 12 | 212 | 121 |
|---|---|---|---|---|---|---|
| $\phi_{1,0}$ | | | | (101) | (1020) | (1020) |
| $\phi'_{1,3}$ | | | | | | |
| $\phi_{2,1}$ | | | | $-1$ | $-(10)$ | $-(10)$ |
| $\phi_{2,2}$ | | (10) | | 1 | | (10) |
| $\phi''_{1,3}$ | | | (10) | 1 | (10) | |
| $\phi_{1,6}$ | 1 | | | | | |

| | 2121, 1212 | 21212 | 12121 | $w_\circ$ |
|---|---|---|---|---|
| $\phi_{1,0}$ | (10202) | (102020) | (102020) | (1020202) |
| $\phi'_{1,3}$ | 1 | (10) | (10) | |
| $\phi_{2,1}$ | $-1$ | | | |
| $\phi_{2,2}$ | | | $-(10)$ | |
| $\phi''_{1,3}$ | 1 | | (10) | |
| $\phi_{1,6}$ | | | | |

Above, $(a_m \cdots a_1 a_0) := a_0 + \sum_{i=1}^m a_i(q^{i/2} + q^{-i/2})$.

What properties of $z$ or $\psi$ ensure that $\alpha^z_{\psi,G}$ has positive unimodal coefficients?

**Thm (T. 2026)** For a Levi $G' \subseteq G$ with $W' \subseteq W$:

If $z \in W'$, then

$$\text{the } \alpha^z_{\psi,G} \text{ for } \psi \in \mathrm{Irr}(W)$$

are $\mathbf{Z}_{\geq 0}$-linear combinations of

$$\text{the } \alpha^z_{\psi',G'} \text{ for } \psi' \in \mathrm{Irr}(W').$$

**Cor (T. 2026)**

If $z \in W' \subseteq W$, and positivity and unimodality hold for the $\alpha^z_{\psi',G'}$, then they hold for the $\alpha^z_{\psi,G}$.

**Proof**

$\mathrm{IH}^{2i}(\bar{Y}_{z,g})$ and $spr_\psi$ enjoy the same compatibility with induction from $W'$ to $W$.

**Conj (T. 2026)**

Suppose that $\psi \in \mathrm{Irr}(W)$ is inflated from a product of symmetric groups.

Then for any $z \in W$, the nonzero coefficients of $\alpha^z_{\psi,G}$ all have the same sign and are unimodal up to sign.

If the inflation morphism does not descend to

$$W(G_2) \to W(A_2) = S_3,$$

then the sign is positive.

I checked this

- up through rank 5,
- in rank 6 for rationally smooth $z$,
- in rank 7 for rationally smooth $z$ of length $\leq 24$.

Example Type $F_4$.

| | | unimodal | positive | inflated |
|---|---|---|---|---|
| $\phi_{1,0}$ | $F_4$ | | | ✓ |
| $\phi'_{2,4}$ | $F_4(a_1)(11)$ | | | ✓ |
| $\phi_{4,1}$ | $F_4(a_1)$ | × | × | |
| $\phi''_{2,4}$ | $F_4(a_2)(11)$ | | | ✓ |
| $\phi_{9,2}$ | $F_4(a_2)$ | × | × | |
| $\phi'_{8,3}$ | $C_3$ | | | |
| $\phi''_{8,3}$ | $B_3$ | | | |
| $\phi'_{1,12}$ | $F_4(a_3)(211)$ | | | ✓ |
| $\phi''_{6,6}$ | $F_4(a_3)(22)$ | | | |
| $\phi'_{9,6}$ | $F_4(a_3)(31)$ | | × | |
| $\phi_{12,4}$ | $F_4(a_3)$ | | × | |
| $\phi'_{4,7}$ | $C_3(a_1)(11)$ | | × | |
| $\phi_{16,5}$ | $C_3(a_1)$ | | × | |
| $\phi'_{6,6}$ | $\tilde{A}_2 + A_1$ | | × | |
| $\phi_{4,8}$ | $B_2(11)$ | | | ✓ |
| $\phi''_{9,6}$ | $B_2$ | | × | |
| $\phi''_{4,7}$ | $A_2 + \tilde{A}_1$ | | × | |
| $\phi'_{8,9}$ | $\tilde{A}_2$ | | | |
| $\phi''_{1,12}$ | $A_2(11)$ | | | ✓ |
| $\phi''_{8,9}$ | $A_2$ | | | |
| $\phi_{9,10}$ | $A_1 + \tilde{A}_1$ | | | |
| $\phi'_{2,16}$ | $\tilde{A}_1(11)$ | | | ✓ |
| $\phi_{4,13}$ | $\tilde{A}_1$ | | × | |
| $\phi''_{2,16}$ | $A_1$ | | | ✓ |
| $\phi_{1,24}$ | $1$ | | | ✓ |

Webpage with data and accompanying code: https://mqtrinh.github.io/math/research/code/chevie/

Claude Sonnet 4.6 helped me write the regex scripts.

*Thank you for listening.*