WHAT GAUSS KNEW ABOUT KNOTS AND BRAIDS

2021 MIT IAP MATHEMATICS LECTURE SERIES

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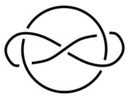
Problem 1. Show that the figure-8 knot is not tricolor.

Problem 2. Read about the diagrammatic definition of the linking number. Show that the link below, appropriately oriented, has linking number zero:



Problem 3. Show that for all $n \geq 2$, there is a braid on n strands whose closure is the unknot. It is *not* the identity.

Problem 4. Let σ_1 denote the generator of Br_2.

(1) Show that the HOMFLY invariants $\mathbf{P}(\widehat{\sigma_1^n})$ satisfy a linear recurrence in n.
(2) Deduce that the $(2, n)$-torus links are pairwise non-isotopic.

Problem 5. Show that in Br_4, the elements

$$(\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_1^7 \quad \text{and} \quad (\sigma_1\sigma_2\sigma_3)^6\sigma_1$$

have the same link closure. How would you generalize this observation?

References

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